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Forecasting Value-at-Risk for equity returns with regime switching models

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Abstract

We use an univariate and multivariate Markov-Switching model to capture regime structures of three different equity returns. Besides a Markov-Switching approach, we use an endogenous regime switching model to capture the regime dynamics of equity returns. For each equity return we forecast the Value-at-Risk with a horizon of one month. We find that both Markov-Switching and endogenous regime switching models identify states with high volatility and a low intercept, and states with low volatility and a high intercept, except for the emerging market equity where no significant lower intercept is present in the high volatility state. Endogenous regime switching is present in developed market equity and private equity, indicating leverage effects in developed market equity and both leverage and trend following effects in private equity. Only the forecasts obtained from the multivariate Markov-Switching model pass the calibration backtests for all equities.

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1 Introduction

Banks, insurance companies and pension funds invest in portfolios containing different equity assets. Therefore it is important to assess the risk of equities. In this thesis we use a Markov-Switching (MS) model to capture possible regime structures in the conditional mean and conditional variance of equity returns. We consider developed market equity, emerging market equity and private equity in both univariate and multivariate settings. Besides the MS model, the endogenous regime switching (ERS) model of Chang et al. (2017) will be used for the univariate setting where a latent variable drives the transition of the states such that the transition of the regimes is allowed to be correlated with observable variables. We compare these models based on their performance for one month ahead Value-at-Risk (VaR) forecasts.

We are interested in the following three research questions. First, for both univariate and multivariate models, are there regime switching structures present in the different equity returns? Second, can the ERS model capture different equity dynamics than the MS models in our empirical analysis? Third, do ERS models help to improve the univariate VaR forecast?

We use a MS Vector Autoregression (MSVAR) model for the multivariate setting and a MS Autoregression (MSAR) model for the univariate setting. In contrast to the MS models, the ERS model has time-varying transition probabilities. We follow the approach of Chang et al. (2017) where they assume a latent variable that controls the regime switching based on a threshold parameter. The error term of the observation equation is allowed to be correlated with the error term of the latent variable in the next period. For all models we assume two regimes or states. We evaluate the models based on the two-step procedure of Nolde and Ziegel (2017). In the first step individual models are tested for correct calibration of the forecasts, similar to traditional unconditional and conditional tests. When the first stage is passed, the second step compares the accuracy between the different models based on a comparative backtest.

Different studies attempt to detect possible regimes in the stock market returns. For example, Maheu and McCurdy (2000) use a MS model with time-varying transition probabilities depending on a duration measure to identify regimes. They are able to identify states with high returns and low volatility, and states with low returns and high volatility.

Hamilton and Lin (1996) use a bivariate MS model for monthly stock returns and growth in industrial production. They indicate that economic recessions play an important role in driving the volatility of stock returns. Additionally, their bivariate model is able to forecast economic turning points. In our research we do not use a duration measure to identify regimes, but we incorporate endogenous switching to identify regimes. Also we compare the ERS model with the MS(V)AR models for their performance in VaR forecasting.

Several studies applied the endogenous regime switching see, for example, Kim et al. (2008) and Chang et al. (2017). Chang et al. (2017) indicate that endogenous switching is present in their empirical research in GDP growth rate. They also suggest that the link between the latent variable and the independent variable might contain important information which is not captured when using the standard MS model. Most importantly, the correlation between the shock to the observerved time series and the shock to the latent variable allows a different way of manifesting mean reverting and leverage effects.

There are several studies using the MS property for forecasting VaR. For example, Ardia et al. (2018) investigate whether a combination of the MS and the GARCH model may improve the risk forecasts of securities typically held by fund managers. They find that MS GARCH models deliver better VaR forecasts, expected shortfall, and left-tail distribution forecasts than their single-regime models, this especially holds for stock return data. Moreover, these improvements are more pronounced when the MS property is applied to simple specifications such as the GARCH-Normal model.

Sajjad et al. (2008) indicate that the MS GARCH-t model outperforms other GARCH family models in forecasting one day ahead VaR for the FTSE100 index. For the other S&P500 index, the MSGARCH-t and EGARCH-t models outperform the other models. Our research differs in at least three aspects from the papers mentioned above. First, we also take a multivariate approach instead of an univariate approach with GARCH specifications. Second, we also allow endogenous switching in our univariate model. Lastly, the forecast horizon in our research is not one day or one week, but one month.

We find that both MSAR and ERS models characterise the crisis state or state 1 for emerging market equity, as high volatility with trend following effects compared to the normal state or state 0. For developed market equity, state 1 also has a significant lower intercept than state 0 but there are no trend following effects. For private equity, state 1 has both a significant lower intercept and a trend following effect. The MSVAR model characterises state 1 similarly, except there are no trend following effects and the covariance between the error terms is higher in state 1. The correlation parameter of the endogenous regime switching is significant around -0.99 for developed market equity and private equity, indicating leverage effects in both equities and also a trend following effect in private equity. Therefore the ERS model is able to capture additional dynamics, but the ERS model does not improve the VaR forecasts in terms of accuracy compared to the other models in our analysis. Only the MSVAR model is not rejected by either calibration backtests for all equity returns.

2 Data

We consider three equity assets: developed market equity, emerging market equity and private equity. We use the following corresponding proxies respresenting performance in Europe: MSCI Europe Index, MSCI Emerging Markets Index and the LPX Listed Private Equity Index Series. For all data series, the period from 2000 March until 2019 June will be used.

We calculate the continuously compounded returns for each series. Table 1 shows the summary statistics of the index returns. The MSCI Emerging Market Index return has the highest standard deviation of 7.58%. The LPX Listed Private Equity Index return is more volatile than the MSCI Europe Index return with standard deviations 6.04% and 4.32%, respectively. All return series exhibit negative skewness and have higher skewness than 3.

Table 1: Summary statistics of the equity index returns in percentage.

Index	Mean	Standard deviation	Kurtosis	Skewness
MSCI Emerging Market Europe	0.08	7.58	4.67	-0.68
MSCI Europe	-0.02	4.32	4.07	-0.70
LPX Listed Private Equity	0.10	6.04	8.10	-1.05

Note: Sample is from 2000 March to 2019 June.

3 Methodology

3.1 MSVAR model

We use both univariate MSAR and multivariate MSVAR models. Only the multivariate approach will be shown, because the univariate approach is applied in a similar way. Let us denote $\boldsymbol{y_t}$ as the vector representing the three equity returns at time t.

Consider the following MSVAR model:

$$\boldsymbol{y_t} = \boldsymbol{\alpha}_{S_t} + H_{S_t}\boldsymbol{y_{t-1}} + \boldsymbol{q_t}, \qquad \boldsymbol{q_t} \sim N(0, \Sigma_{S_t}),$$

where the intercept and loading parameters depend on state S_t. The error terms of $\boldsymbol{q_t}$ are assumed to be normally distributed with expectation zero and state dependent covariance matrix Σ_{S_t}. The state transitions follow a Markov chain of first order. We define $p_{ij} = P(S_t = i|S_{t-1} = j)$ where $S_t \in \{0, 1\}$.

3.2 Endogenous regime switching model

We consider the univariate ERS model of Chang et al. (2017), where a latent variable $w_{l,t}$ determines the transition of the states of asset l. The latent variable is defined as

$$w_{l,t} = \alpha w_{l,t-1} + v_{l,t}, \qquad v_{l,t} \sim N(0, 1).$$

The state is then defined as

$$S_{l,t} = \mathbb{1}(w_{l,t} \geq \tau_l),$$

where $\mathbb{1}(.)$ is the indicator function and τ_l is a threshold parameter. The state $S_{l,t}$ here is defined in a different way compared to the state in the MS(V)AR models, but we use the same notation for simplicity. Consider the univariate setup for the return of asset l:

$$y_{l,t} = m_{l,t} + \sigma_{l,t} u_{l,t},$$

where $m_{l,t} = m(y_{t-1}, w_{l,t}; \boldsymbol{\Theta_l}) = m_{l,S_t} = a_{l,S_t} + F_{l,S_t} y_{t-1}$ and $\sigma_{l,t} = \sigma(w_{l,t}; \boldsymbol{\Theta_l}) = \sigma_{l,S_t}$, are the conditional mean and conditional standard deviation respectively. $\boldsymbol{\Theta_l}$ denotes the parameters for the univariate model of asset l. $u_{l,t}$ is the error term assumed to be normally distributed and correlated with $v_{l,t}$ as

$$\begin{pmatrix} u_{l,t} \\ v_{l,t+1} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_l \\ \rho_l & 1 \end{pmatrix} \right).$$

We leave the index $l \in \{1, 2, 3\}$ out for convenience in the remainder of this thesis. Note that when ρ is restricted to zero, the state transitions become a first order Markov Chain. There is a unique pair of τ and α which corresponds to a specific set of transition probabilities p_{11} and p_{22} when $|\alpha| \leq 1$ as shown in Lemma 2.1 of Chang et al. (2017). This implies that the ERS model becomes observationally equivalent to the MSAR model.

3.3 Estimation of the MSVAR model

We use the Expectation-Maximisation (EM) algorithm to estimate the parameters. For the expectation step, the filter and smoother of Hamilton (1989) are used. We take the expectation with respect to the regimes given all information of the estimation sample. For the maximisation step, we obtain analytical results. The algorithm is run until convergence of the parameters.

For the MSVAR model, using the law of conditional probability, we obtain the logarithm of the likelihood function as

$$\log p(\boldsymbol{y_1}, \ldots, \boldsymbol{y_T}|\boldsymbol{\theta}) = \log p(\boldsymbol{y_1}, \ldots, \boldsymbol{y_T}, S_1, \ldots, S_T|\boldsymbol{\theta}) - \log p(S_1, \ldots, S_T|\boldsymbol{y_1}, \ldots, \boldsymbol{y_T}; \boldsymbol{\theta}),$$

where $\boldsymbol{\theta}$ contains the parameters of the intercepts, covariance matrices, loading matrices, probability matrix and the initial state (at $t = 0$) of the estimation sample. In the expectation step we take the expectation of this likelihood with respect to the states given $\boldsymbol{y_1}, \ldots, \boldsymbol{y_T}$. For convenience we use the following expectation operator: $\tilde{E}[.] := E_{S_{1:T}|y_{1:T}}[.]$ where $S_{1:T} := (S_1, \ldots, S_T)$ and $y_{1:T} := (\boldsymbol{y_1}, \ldots, \boldsymbol{y_T})$. The location of $\boldsymbol{\theta}$ for the maximum value of $\log p(y_{1:T}|\boldsymbol{\theta})$ has the same location $\boldsymbol{\theta}$ for the maximum value of $\log p(y_{1:T}, S_{1:T}|\boldsymbol{\theta})$ because $\tilde{E}[\frac{d}{d\boldsymbol{\theta}} \log p(S_{1:T}|y_{1:T}; \boldsymbol{\theta})] = 0$. This can be shown by integrating out the states. Therefore we maximize the expectation of the joint log-likelihood function:

$$\tilde{E}[\log p(y_{1:T}, S_{1:T}|\boldsymbol{\theta})] = \sum_{t=1}^{T} \left\{ \sum_{i,j=1}^{n} p^*_{ij}(t) \log[f_i(\boldsymbol{y_t}) p_{ij}] \right\} + \sum_{j=1}^{n} p^*_j(0) \rho_j,$$

where $f_k(\boldsymbol{y_t}) = f(\boldsymbol{y_t}|S_t = k; \boldsymbol{\theta_k})$ is the normal probability density function with mean and covariance matrix in state k, $p^*_{ij}(t) = P(S_t = i, S_{t-1} = j|\mathcal{F}_T)$ are the smoothed transition probabilities obtained by the Hamilton Smoother, $p^*_j(t) = P(S_t = j|\mathcal{F}_T)$, $\rho_j = P(S_0 = j)$, $\boldsymbol{\theta_k}$ is the parameter vector including the intercept, loadings matrix and covariancematrix when $S_t = k$, and $\mathcal{F}_T$ is the information set including $\{\boldsymbol{y_1}, \ldots, \boldsymbol{y_T}\}$. Note that $p^*_k(t) = p^*_{k1}(t) + p^*_{k2}(t)$. To optimize the expectation of the joint log-likelihood with respect to the parameters $\boldsymbol{\alpha}_{S_t}$, H_{S_t} and Σ_{S_t}, note that

$$\frac{d\tilde{E}[\log p(y_{1:T}, S_{1:T})]}{d\boldsymbol{\theta}_k} = 0 \Leftrightarrow \sum_{t=1}^{T} \sum_{i,j=1}^{n} \frac{dp^*_{ij}(t) \log[f_i(\boldsymbol{y_t})p_{ij}]}{d\boldsymbol{\theta_k}} = 0 \tag{1}$$

$$\Leftrightarrow \sum_{t=1}^{T} \sum_{j=1}^{n} p^*_{kj}(t) \frac{dlog[f_k(\boldsymbol{y_t})]}{d\boldsymbol{\theta_k}} = 0 \Leftrightarrow \sum_{t=1}^{T} p^*_k(t) \frac{d \log[f_k(\boldsymbol{y_t})]}{d\boldsymbol{\theta_k}} = 0. \tag{2}$$

The optimization with respect to the intercept, loading matrix and covariance matrix gives standard results, namely the weighed least squares. For the derivation see the Appendix.

3.4 Estimation of the endogenous regime switching model

We use the Maximum Likelihood method along the lines of Chang et al. (2017) to estimate the ERS model. The standard Hamilton filter cannot be used since the error term of the observation is assumed to be correlated with the error term of the latent variable in the next period. For the ERS model we only consider the univariate setting. Consider the predictive likelihood shown below where we ignore the index l of a specific asset:

$$L(\boldsymbol{\Theta}) = \prod_{t=1}^{T} p(y_t|\mathcal{I}_{t-1}; \boldsymbol{\Theta}),$$

where $\boldsymbol{\Theta}$ is the parameter vector of the model and $\mathcal{I}_{t-1}$ is the information set including $\{y_1, \ldots, y_{t-1}\}$. Note that

$$p(y_t|\mathcal{I}_{t-1}; \boldsymbol{\Theta}) = \sum_{S_t} p(y_t|S_t, \mathcal{I}_{t-1}; \boldsymbol{\Theta}) p(S_t|\mathcal{I}_{t-1}; \boldsymbol{\Theta}),$$

by the conditional law of probabilities and the law of total probability. We obtain $p(y_t|S_t, \mathcal{I}_{t-1}; \boldsymbol{\Theta})$ by computing the normal probability density function $N(m_t, \sigma_t^2)$ evaluated at y_t where m_t is the conditional mean and σ_t^2 is the conditional variance given S_t and $\mathcal{I}_{t-1}$ as specified in

Subsection 3.2. For $p(S_t|\mathcal{I}_{t-1};\mathbf{\Theta})$ we need the modified filter of Chang et al. (2017). The prediction equation is shown below:

$$p(S_t|\mathcal{I}_{t-1};\mathbf{\Theta}) = \sum_{S_{t-1}} p(S_t|S_{t-1},\mathcal{I}_{t-1};\mathbf{\Theta})p(S_{t-1}|\mathcal{I}_{t-1};\mathbf{\Theta}).$$

Here $p(S_t|S_{t-1},\mathcal{I}_{t-1};\mathbf{\Theta}) = p(S_t|S_{t-1},y_{t-1},y_{t-2};\mathbf{\Theta}) = (1-S_t)w_p + S_t(1-w_p)$ where w_p is the probability to go to the low state conditional on the information of $(S_{t-1},y_{t-1},y_{t-2})$ which is derived in Chang et al. (2017). The derivation is based on the law of conditional probabilities and using the cumulative distribution function of the standard normal distribution. If $|\rho|< 1$ and $|\alpha|< 1$, then

$$w_p = \frac{\left[(1-S_{t-1})\int_{-\infty}^{\tau\sqrt{1-\alpha^2}} + S_{t-1}\int_{\tau\sqrt{1-\alpha^2}}^{\infty}\right]\Phi\left\{(\tau-\rho u_{t-1}-\frac{\alpha x}{\sqrt{1-\alpha^2}})/(\sqrt{1-\rho^2})\right\}\phi(x)dx}{(1-S_{t-1})\Phi(\tau\sqrt{1-\alpha^2}) + S_{t-1}\left[1-\Phi(\tau\sqrt{1-\alpha^2})\right]},$$

where Φ is the cumulative distribution function of the standard normal distribution and ϕ is the probability density function of the standard normal distribution. In our context u_{t-1} is estimated as $\frac{y_{t-1}-\hat{a}_{S_{t-1}}-\hat{F}_{S_{t-1}}y_{t-2}}{\hat{\sigma}_{S_{t-1}}}$ and $p(S_{t-1}|\mathcal{I}_{t-1};\mathbf{\Theta})$ is obtained from the updating equation:

$$p(S_t|\mathcal{I}_t;\mathbf{\Theta}) = p(S_t|y_t,\mathcal{I}_{t-1};\mathbf{\Theta}) = \frac{p(y_t|S_t,\mathcal{I}_{t-1};\mathbf{\Theta})p(S_t|\mathcal{I}_{t-1};\mathbf{\Theta})}{p(y_t|\mathcal{I}_{t-1};\mathbf{\Theta})}.$$

With the probabilities w_p we can obtain the transition probabilities $P(S_t = i|S_{t-1} = j, y_{t-1}, y_{t-2})$.

With an initialisation at $t = 1$ for $p(S_t|\mathcal{I}_t;\mathbf{\Theta})$ and the parameters we can iteratively calculate the values of the prediction step, updating step and so forth. For parameters ρ and τ we use specific multiple initialisations, -0.4 and 0.4 for ρ, and -0.4 and 0.8 for τ. After iterating over the estimation sample, we use maximum likelihood with respect to the logarithm of $L(\mathbf{\Theta})$ to obtain the parameters.

3.5 VaR Forecasts

For the MS(V)AR models we only show the multivariate approach to forecast, since the univariate approach is applied in a similar way. To forecast the VaR, we forecast the mean and covariance matrix of the conditional distribution $p(\boldsymbol{y_{t+1|t}}|\boldsymbol{\theta})$. The forecasted mean of this distribution is also the forecast of $\boldsymbol{y_{t+1|t}}$. We simulate from an estimation of this conditional distribution to obtain VaR forecasts. For the MSVAR model let us first define $\hat{\boldsymbol{\xi}}_{t|t} := [\hat{P}(S_t = 0|\mathcal{F}_t), \hat{P}(S_t = 1|\mathcal{F}_t)]'$ which is obtained from the recursive Hamilton filter. The point forecasts are constructed as

$$\hat{\boldsymbol{y}}_{t+1|t} = \hat{P}(S_{t+1} = 0|\mathcal{F}_t) * (\hat{\boldsymbol{\alpha}}_0 + \hat{H}_0 \boldsymbol{y_t}) + \hat{P}(S_{t+1} = 1|\mathcal{F}_t) * (\hat{\boldsymbol{\alpha}}_1 + \hat{H}_1 \boldsymbol{y_t}),$$

where $\hat{P}(S_{t+1} = k|\mathcal{F}_t)$ is obtained by: $\hat{\boldsymbol{\xi}}_{t+1|t} = \hat{P} * \hat{\boldsymbol{\xi}}_{t|t}$. The matrix $\hat{P}$ contains the estimated transition probabilities $\hat{p}_{ij}$. The estimated distribution of the error term vector $\hat{\boldsymbol{q}}_{t+1|t}$ is a linear combination of the normal distributions $\boldsymbol{e_0} \sim N(\boldsymbol{0}, \hat{\Sigma}_0)$ and $\boldsymbol{e_1} \sim N(\boldsymbol{0}, \hat{\Sigma}_1)$:

$$\hat{\boldsymbol{q}}_{t+1|t} \overset{d}{\sim} \hat{P}(S_{t+1} = 0|\mathcal{F}_t)\boldsymbol{e_0} + \hat{P}(S_{t+1} = 1|\mathcal{F}_t)\boldsymbol{e_1}.$$

By simulating the error term $\hat{\boldsymbol{q}}_{t+1|t}$ L times, we obtain simulated values of $\hat{\boldsymbol{y}}_{t+1|t}$. Then we forecast the VaR by taking the $(1-\alpha)$ quantile from the L simulated values of $\hat{\boldsymbol{y}}_{t+1|t}$. We take $L = 100000$ and $\alpha = 0.975$.

A similar approach is used for the ERS model. Recall that the states are different for each asset in this model, but we leave the index l out for convenience and use the same notation S_t for the state. The important difference is the different prediction filter used to compute $\hat{P}(S_{t+1} = 1|\mathcal{I}_t)$ and $\hat{P}(S_{t+1} = 0|\mathcal{I}_t)$. See the prediction step in Subsection 3.4 for this computation. The point forecasts for the univariate assets are constructed as

$$\hat{y}_{t+1} = \hat{P}(S_{t+1} = 0|\mathcal{I}_t)(\hat{a}_0 + \hat{F}_0 y_t) + \hat{P}(S_{t+1} = 1|\mathcal{I}_t)(\hat{a}_1 + \hat{F}_1 y_t).$$

The estimated distribution of the error term $\sigma_{t+1}u_{t+1}$ is also a linear combination of normal distributions:

$$\hat{P}(S_{t+1} = 0|\mathcal{I}_t)\epsilon_0 + \hat{P}(S_{t+1} = 1|\mathcal{I}_t)\epsilon_1,$$

where $\epsilon_0 \sim N(0, \hat{\sigma}_0^2)$ and $\epsilon_1 \sim N(0, \hat{\sigma}_1^2)$. Again we simulate the estimated error term L times and and obtain the simulated $\hat{y}_{t+1|t}$. Then we forecast the VaR by taking the $(1-\alpha)$ quantile of the L simulated values of $\hat{y}_{t+1|t}$.

3.6 VaR Evaluation

We split our data into two parts. An initial estimation window with E months and an out-of-sample period with P months. For each month $t \in \{E+1, \dots, E+P\}$ a VaR forecast over one month is constructed using the observations in the past E months. We take $E = 90$ and $P = 142$.

A moving window is used for all three models. The evaluation procedure consists of two stages along the lines of Nolde and Ziegel (2017). In the first step, we test calibration of individual models with traditional backtests. For both unconditional and conditional calibration tests we use identification functions instead of loss functions as in Nolde and Ziegel (2017). For each model, denote the VaR forecast of R_t by $V\hat{a}R(R_t)$. The test statistic for the unconditional calibration is

$$\mathcal{T}_1 = \frac{1}{\sqrt{P}} \sum_{t=E+1}^{E+P} G(V\hat{a}R(R_t), R_t)\hat{\Sigma}_P^{-1/2},$$

where $G(V\hat{a}R(R_t), R_t) = 1 - \alpha - \mathbb{1}(R_t < V\hat{a}R(R_t))$ is the identification function of the elicitable risk measure VaR, $\hat{\Sigma}_P$ is the heteroscedasticity and autocorrelation consistent (HAC) estimator of the asymptotic variance $\Sigma_P = cov(\frac{1}{\sqrt{P}} \sum_{t=E+1}^{E+P} G(V\hat{a}R(R_t), R_t))$, see Nolde and Ziegel (2017) for details. The null hypothesis is that the expectation of the identification function for every t is 0. As $P \to \infty$ the test statistic converges to the standard normal distribution under the null hypothesis and certain conditions, see Theorem 4 of Giacomini and White (2006) for details.

The conditional calibration test in our setting is

$$\mathcal{T}_2 = P\Big\{\frac{1}{P} \sum_{t=E+1}^{E+P} G(V\hat{a}R(R_t), R_t)\boldsymbol{h_t}\Big\}'\hat{\Omega}_P^{-1}\Big\{\frac{1}{P} \sum_{t=E+1}^{E+P} G(V\hat{a}R(R_t), R_t)\boldsymbol{h_t}\Big\},$$

where we take $\boldsymbol{h_t} = (1, V\hat{a}R(R_t))'$ with dimension $q = 2$. $\boldsymbol{h_t}$ is also referred to as the test functions which should represent the most important information available at time $t-1$. The idea is that given the information set of $t-1$, the expectation of the identification function at time t is 0 under the null hypothesis. Note that the notation differs from the general conditional calibration test of Nolde and Ziegel (2017) because they use a matrix of test functions for multiple risk measures. $\hat{\Omega}_P$ is the consistent estimator of the covariance matrix of $G(V\hat{a}R(R_t), R_t)\boldsymbol{h_t}$. Under the null hypothesis the asymptotic distribution of this test is the $\chi^2(q)$ distribution, see Theorem 1 of Giacomini and White (2006).

In step two, we use two null hypotheses for comparing accuracy. H_0^-: model $m1$ predicts at least as well as model $m2$, and H_0^+: model $m1$ predicts at most as well as model $m2$. Define the following two quantities as in Nolde and Ziegel (2017):

$$\zeta^* := \limsup_{P\to\infty} \frac{1}{P} \sum_{t=E+1}^{E+P} E(S(V\hat{a}R_{m1}(R_t), R_t) - S(V\hat{a}R_{m2}(R_t), R_t)),$$
$$\zeta_* := \liminf_{P\to\infty} \frac{1}{P} \sum_{t=E+1}^{E+P} E(S(V\hat{a}R_{m1}(R_t), R_t) - S(V\hat{a}R_{m2}(R_t), R_t)),$$

where $S(R_t, V\hat{a}R(R_t)) = (1 - \alpha - \mathbb{1}(R_t < V\hat{a}R(R_t)))(-V\hat{a}R(R_t)) + \mathbb{1}(R_t < V\hat{a}R(R_t))(-R_t)$ is the consistent scoring function as in Nolde and Ziegel (2017). There is a small difference in the sign of the score function because their paper works with losses instead of returns. We can reformulate the hypotheses as $H_0^- : \zeta^* \leq 0$ and $H_0^+ : \zeta_* \geq 0$. To test H_0^- or H_0^+, we use a test statistic in a Diebold-Mariano type as

$$\mathcal{T}_3 = \frac{\Delta_P \bar{S}}{\hat{\sigma}_P/\sqrt{P}},$$

where $\Delta_P \bar{S} = \frac{1}{P}\sum_{t=E+1}^{E+P}(S(V\hat{a}R_{m1}(R_t), R_t) - S(V\hat{a}R_{m2}(R_t), R_t))$ and $\hat{\sigma}_P^2$ is an HAC estimator of the asymptotic variance $\sigma_P^2 = var(\sqrt{P}\Delta_P \bar{S})$. Under H_0^- and for large enough P, $\Delta_P \bar{S}$ has expected value less or equal to zero. Under H_0^+ and for large enough P, the expected value is larger or equal to zero. Based under certain conditions shown in Theorem 4 in Giacomini and White (2006), we can reject H_0^+ if $\Phi(\mathcal{T}_3) \leq \eta$ and we can reject H_0^- if $1 - \Phi(\mathcal{T}_3) \leq \eta$ where η is the significance level. We use the three-zone approach suggested by Fissler et al. (2016) to compare the different VaR forecasts. Model $m1$ is in the red region when H_0^- is rejected, in the yellow region when neither H_0^- nor H_0^+ is rejected and in the green region when H_0^+ is rejected. There are no clear preferences in the yellow region.

4 Results

Recall that the ERS model is observationally equivalent to the MSAR model when the correlation parameter ρ is restricted to zero. We will present the parameters of the MSAR model in the functional form of the ERS model, such that the comparison with the ERS model is more convenient. We also test the presence of endogeneity based on the Likelihood-Ratio (LR) test for $\rho = 0$, this test statistic has a chi-square limiting distribution of one degree of freedom. Table 2 shows the parameters of both models and the p-values of the LR test. For the emerging market equity, AR parameter F_1 is significant and positive in both models while the parameters a_0, a_1 and F_0 are not significant. A positive AR parameter suggests a trend following effect, thus the positive F_1 implies a trend following effect in state 1. The volatility parameters σ_0 and σ_1 are significant, around 4.26 and 9.62 respectively. State 1 can be regarded as the crisis period with high volatility including a trend following effect. The persistence parameter α of the latent variable is high around 0.99 in both models and the correlation parameter ρ in the ERS model is not significant . There are several differences for the developed market equity, parameter a_0 is positive and significant in both models while F_1 is not significant. Also the correlation parameter ρ in the ERS model is significant around -0.98. This implies that there is no trend following effect in state 1, but there is a trend following effect via endogenous regime switching, as a negative shock increases the probability to be in state 1 with a lower intercept in the next period and a positive shock increases the probability to be in state 0 with a higher intercept in the next period. Another implication is that leverage effects are present, meaning that a negative error term of the observed time series is more likely to be followed by a volatile period and vice versa. State 1 for developed market equity can be regarded as the crisis period with high volatility with a significant lower intercept than state 0. For private equity, F_1 is positive and significant in both models, parameter ρ in the ERS model is significant around -0.99. Thus there are trend following effects in state 1 and trend following effects via endogenous regime switching. Moreover, F_0 is only significant and positive in the MSAR model, indicating mean-reversion. Therefore a decline in return from the mean in state 0 is expected to increase after and vice versa. State 1 in private equity can be viewed as a crisis period with high volatility including trend following effects and a low intercept. The trend following effect has the opposite pattern of mean-reversion. Therefore the identification of regimes and awareness of different state characteristics across equities can be important for investors to decide when to buy or sell equities. The difference of trend following effects in emerging market equity, private equity and developed market equity could mean that the emerging market equity and private equity are more sensitive to crisis periods compared to the developed market equity. This could explain the higher kurtosis for private equity and emerging market equity compared to developed market equity since more outliers may occur in state 1 because of the trend following effects. Finally, only for developed market equity the LR test is not rejected at the 5% confidence level, indicating the presence of endogenous regime switching.

Table 2: Maximum likelihood parameters of the ERS model and the MSAR model.

Asset	Emerging market equity		Developed market equity		Private equity	
	ERS	MSAR	ERS	MSAR	ERS	MSAR
a_0	0.67 (0.52)	0.65 (0.51)	0.91 (0.29)	1.15 (0.32)	1.15 (0.37)	2.25 (0.39)
a_1	-0.45 (1.07)	-0.42 (1.65)	-1.57 (0.93)	-1.28 (0.80)	-1.53 (1.30)	-2.56 (0.94)
F_0	0.01 (0.11)	0.01 (0.12)	-0.07 (0.09)	-0.13 (0.09)	0.12 (0.08)	-0.12 (0.06)
F_1	0.20 (0.10)	0.58 (0.13)	0.07 (0.15)	0.15 (0.12)	0.37 (0.12)	0.58 (0.15)
σ_0	4.26 (0.49)	4.26 (0.78)	2.75 (0.22)	2.54 (0.28)	3.40 (0.29)	2.84 (0.26)
σ_1	9.62 (0.79)	9.62 (0.98)	6.08 (0.55)	5.59 (0.70)	8.64 (0.94)	6.78 (0.77)
α	0.99 (0.01)	0.99 (0.02)	0.80 (0.22)	0.98 (0.02)	0.97 (0.02)	0.80 (0.10)
ρ	0.06 (0.48)	0	-0.98(0.06)	0	-0.99 (0.01)	0
τ	0.29 (1.99)	0.29 (5.32)	1.75 (1.23)	1.03 (2.29)	2.54 (1.47)	0.60 (0.44)
Log-likelihood	-777.437	-777.444	-636.996	-640.243	-688.033	-688.733
P-value	0.901		0.011		0.237	

Note: The complete sample from 2000 March to 2019 June is used for estimation. Standard errors are in parentheses. The log-likelihoods and the p-values of the LR test for $\rho = 0$ are also included.

Table 3 shows the estimated parameters of the MSVAR model. The intercept parameters of the developed market equity and private equity are significant. These are positive in state 0 and negative in state 1. Therefore these findings are in line with the findings of the MSAR and ERS models, because here the intercept is also only significantly lower in state 1 than state 0 for developed market equity and private equity. Note that the first column of H_0 and the second column of H_1 are both negative, suggesting that all three equities revert from the emerging market equity in state 0 and revert from developed market equity in state 1. However all parameters of H_k are not significant indicating no AR patterns. The covariance parameters are significant in both states. Both the variance and covariance parameters are higher in state 1. Therefore state 1 in the MSVAR model is different compared to the ERS and MSAR models in the sense that there are no trend following effects, and that the covariance between the error terms is higher in state 1. The state dependent covariance finding can be relevant for diversification strategies of investors.

Table 3: Estimated parameters of the MSVAR model.

	$k=0$	$k=1$
α_k	$\begin{pmatrix}0.83(0.51) & 1.05(0.27) & 1.47(0.33)\end{pmatrix}'$	$\begin{pmatrix}-1.48(1.51) & -2.14(0.89) & -2.47(1.15)\end{pmatrix}'$
H_k	$\begin{pmatrix}-0.08(0.10) & 0.21(0.30) & 0.06(0.25)\\ -0.07(0.05) & 0.00(0.14) & 0.01(0.12)\\ -0.08(0.07) & 0.15(0.17) & 0.06(0.14)\end{pmatrix}$	$\begin{pmatrix}0.19(0.22) & -0.39(0.48) & 0.33(0.31)\\ 0.13(0.12) & -0.19(0.27) & 0.08(0.18)\\ 0.28(0.17) & -0.46(0.37) & 0.39(0.24)\end{pmatrix}$
Σ_k	$\begin{pmatrix}30.35(3.76) & 8.79(1.58) & 9.96(1.90)\\ - & 8.13(1.01) & 8.21(1.12)\\ - & - & 11.52(1.50)\end{pmatrix}$	$\begin{pmatrix}104.13(18.46) & 46.48(9.38) & 61.37(12.68)\\ - & 33.65(5.98) & 39.71(7.58)\\ - & - & 62.21(11.16)\end{pmatrix}$
p_{kk}	0.97 (0.02)	0.93 (0.04)

Note: Standard errors are in parentheses. The estimation sample is from 2000 March to 2019 June.

Figure 1 and Figure 2 show the probabilities of being in the high volatility state for private equity obtained from the MSAR and ERS model, respectively. Both models have high probabilities close to 1 in the recession periods 2000-2002 (dot com bubble), 2007-2008 (financial crisis) and 2011-2012 (euro crisis). The probabilities of the ERS model in these periods are more stable compared to the MSAR model. The probabilities also increase in 2006, 2016 and 2019 in both models. The increase in 2006 of the ERS model is lower than the increase in the MSAR model. This pattern also holds for the developed market equity, shown in Figure 9 and 10 in the Appendix. For the emerging market equity, the difference is smaller as ρ is not signifcant, shown in Figure 11 and 12 in the Appendix.

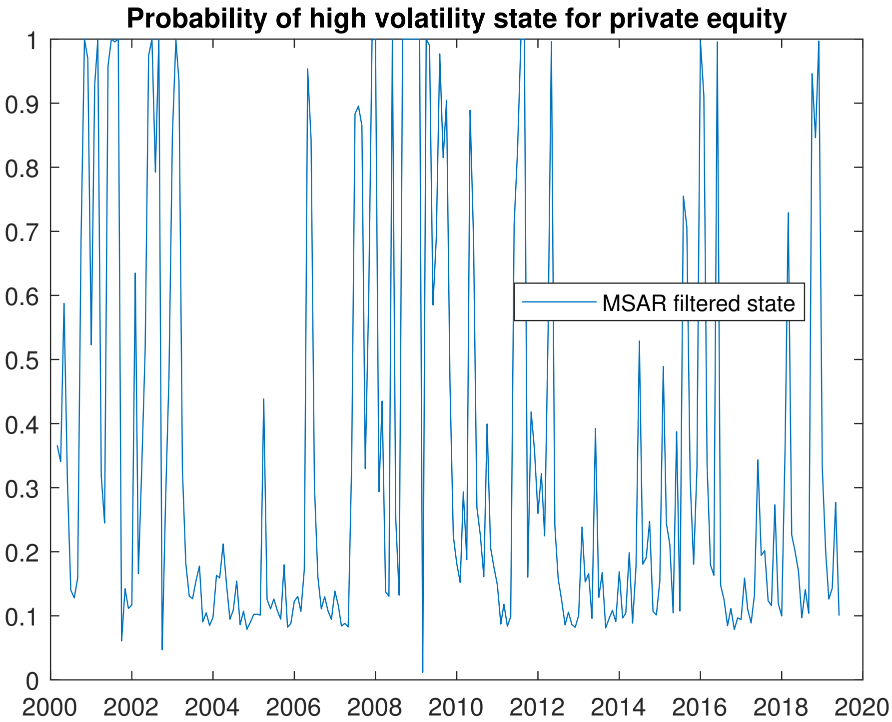

Figure 1: Filtered probabilities of being in high volatility state for private equity obtained from the MSAR model. Sample is from 2000 March to 2019 June.

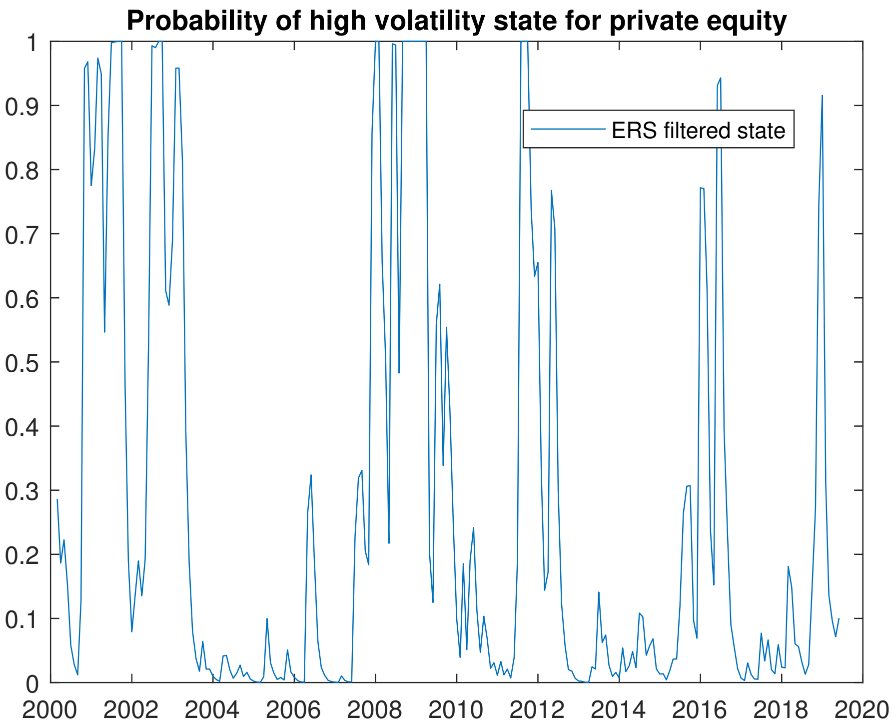

Figure 2: Filtered probabilities of being in high volatility state for private equity obtained from the ERS model. Sample is from 2000 March to 2019 June.

Figure 3 shows the filtered probabilities of being in the high volatility state obtained from the MSVAR. Also the MSVAR model has high probabilities close to 1 in the periods of 2000-2002, 2007-2008, 2011-2012 and 2016. The probabilities during the periods with more stable returns increase less in magnitude compared to the probabilities of the MSAR and ERS models. This indicates that the MSVAR is more conservative in identifying recession periods.

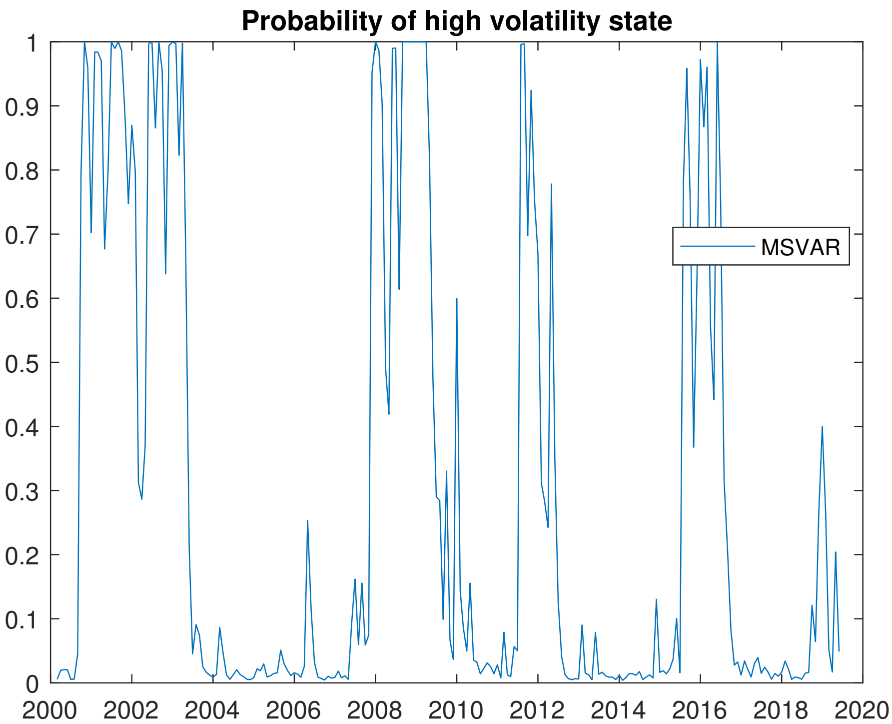

Figure 3: Filtered probabilities of being in high volatility state obtained from the MSVAR model. Sample is from 2000 March to 2019 June.

Figure 4 and Figure 5 show the low-to-low and high-to-high state transition probabilities of all models for private equity, respectively. Both p_{00} and p_{11} of the MSVAR model are higher compared to the MSAR model. The low-to-low transition probability $P(S_t = 0|S_{t-1} = 0, y_{t-1}, y_{t-2})$ of the ERS model changes over time, and decreases sharply in the periods 2001-2002, 2007-2008, 2012 and 2016. The high-to-high transition probability $P(S_t = 1|S_{t-1} = 1, y_{t-1}, y_{t-2})$ of the ERS model is highest on average in the periods 2000-2001 and 2004-2008. The same pattern of the ERS model is observed for the developed market equity, shown in Figure 13 and 14 in the Appendix. For the emerging market equity, the difference in transition probabilities between models is smaller as ρ is not significant, shown in Figure

15 and 16 in the Appendix.

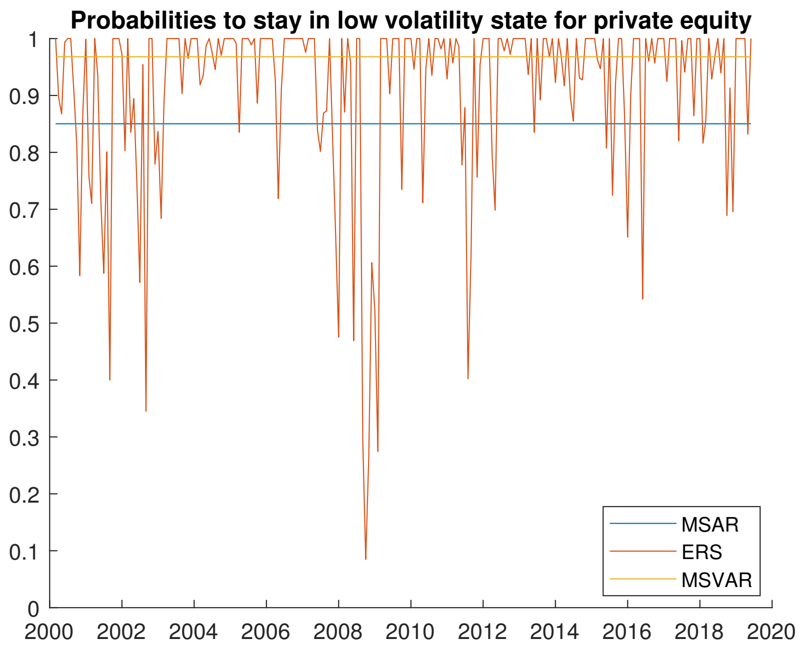

Figure 4: Probabilities to stay in low volatility state for private equity. Sample is from 2000 March to 2019 June.

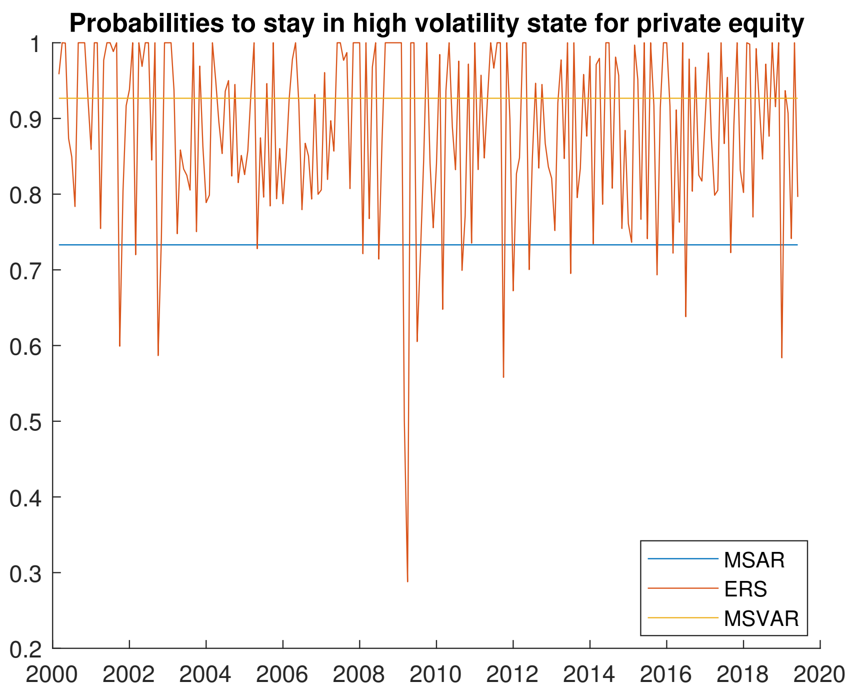

Figure 5: Probabilities to stay in high volatility state for private equity. Sample is from 2000 March to 2019 June.

Table 4 shows the p-values of the conditional and unconditional calibration test. Only the MSVAR model is not rejected by either calibration test for all equity returns. The MSAR model is rejected by the conditional calibration test for the developed market equity and rejected by the unconditional calibration test for private equity at the 5% confidence level. Thus for developed market equity, the information at time $t-1$ might be correlated with the expectation of the identification function at time t as the unconditional calibration is correct. In contrast, the MSAR model does not have correct unconditional calibration for private equity. The ERS model is rejected by both calibration tests for developed market equity and private equity. Therefore the ERS model does not have correct calibration, neither conditionally nor on average. The MSAR and MSVAR models have better unconditional calibration than the ERS model for all equities. An explanation for this finding is that the ERS model might not be capable of predicting due to estimation error, or due to model misspecification.

Table 4: P-values of both unconditional and conditional calibration test.

	Unconditional			Conditional		
Model	MSAR	MSVAR	ERS	MSAR	MSVAR	ERS
Emerging market equity	0.347	0.208	0.102	0.585	0.301	0.145
Developed market equity	0.173	0.121	0.004	0.002	0.229	0.008
Private equity	0.047	0.179	0.009	0.063	0.254	0.023

Note: For each model a moving window of 90 observations is used to forecast. The out of sample is from 2007 September to 2019 June.

Figure 6, Figure 7 and Figure 8 show the scores of the VaR forecasts as defined in Subsection 3.6 for emerging market equity, developed market equity and private equity, respectively. For the emerging market equity, most peaks of the different models occur at the same periods. The highest peaks are in 2008, where the peak of the ERS model is higher than the other models. The MSVAR model has no peaks in 2007 and at the start of 2011 in contrast to the other models. Furthermore, the MSVAR model has higher peaks than the other models at the end of 2011 and 2013. Therefore the MSVAR model seems to perform better in 2007 and at the start of 2011 compared to the other models but worse at the end of 2011 and 2013. For developed market equity, most peaks of the different models do not occur at the same period. The MSVAR model has a low peak in 2007 compared to the other models and it has a peak larger than 6 in 2008 while the other models do not have peaks higher than 1 in 2008. Again the MSVAR model has on average lower peaks in the period 2007-2008 and a lower peak at the start of 2011 compared to the other models but it has a larger peak at the end of 2011. Lastly for private equity, the MSVAR model has the largest peak in 2008, but after 2010 it has fewer and lower peaks on average compared to the other models. The ERS model has the lowest peaks in the period 2007-2008, but after 2008 it has more peaks than the other models. Therefore the ERS model seems to perform better in the period 2007-2008 compared to the other models, but after 2010 the MSVAR model performs better than the other models.

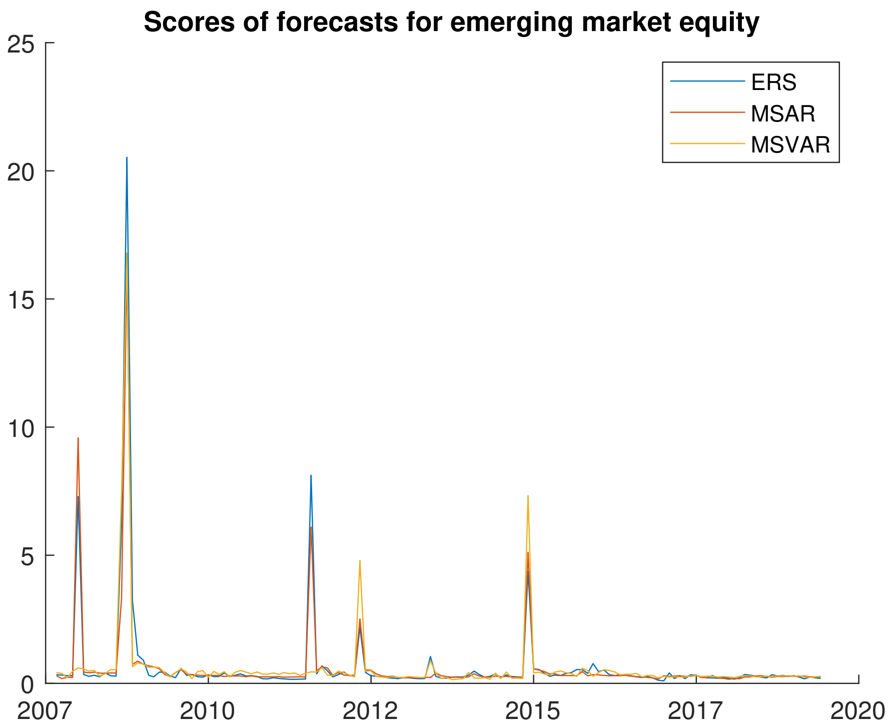

Figure 6: Scores of VaR forecasts for emerging market equity. Sample is from 2007 September to 2019 June.

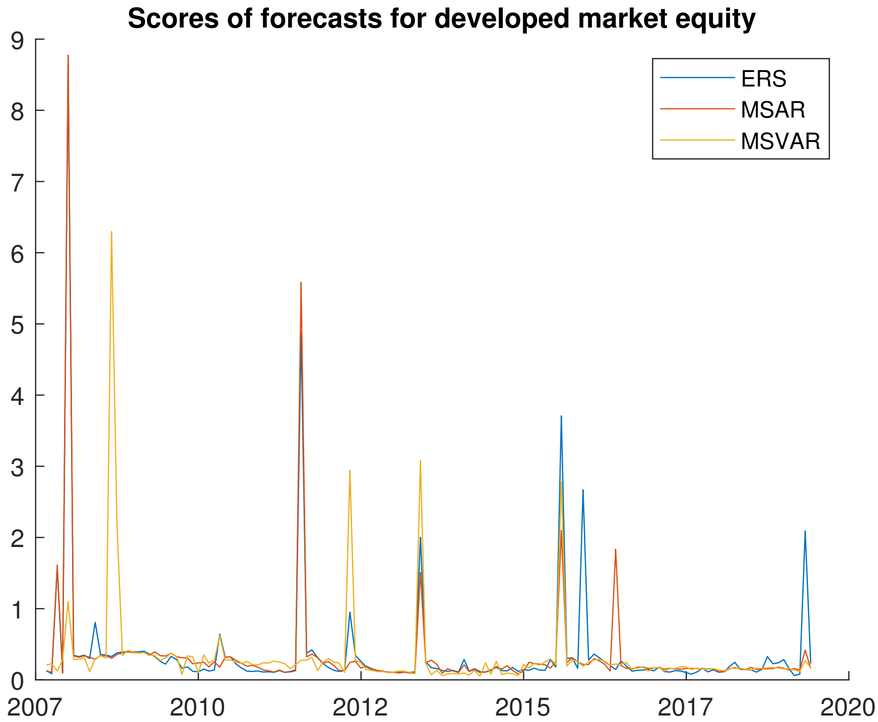

Figure 7: Scores of VaR forecasts for developed market equity. Sample is from 2007 September to 2019 June.

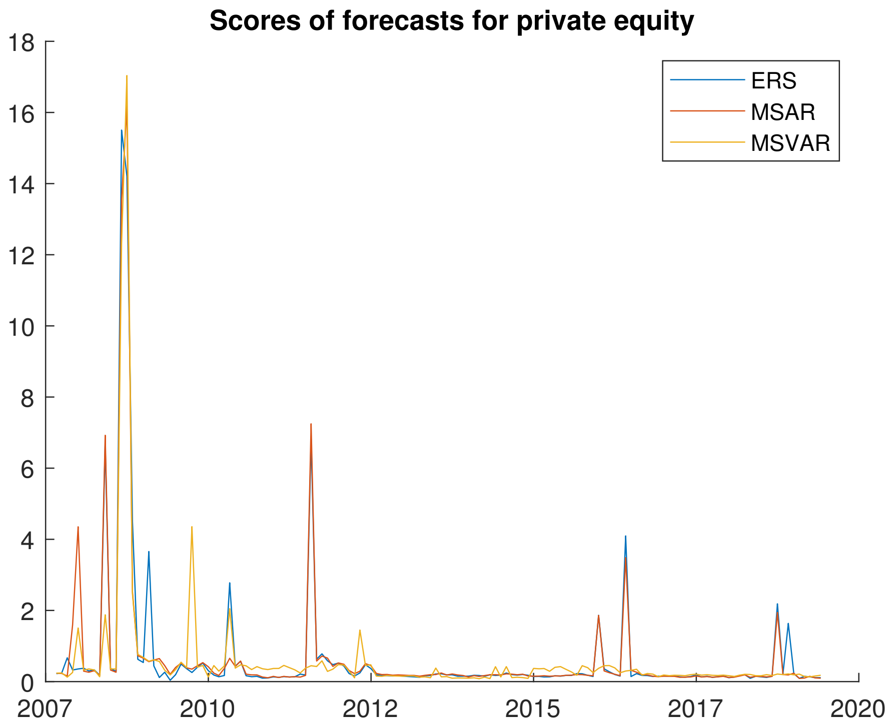

Figure 8: Scores of VaR forecasts for private equity. Sample is from 2007 September to 2019 June.

Table 5 shows the p-values of hypothesis H_0^- of the comparative backtest which is that model $m1$ predicts at least as well as model $m2$. In the comparative test for all equities, both hypotheses H_0^- and H_0^+ cannot be rejected. Recall that this is the yellow region, meaning that there is no clear difference in terms of accuracy among the models. Note that all p-values are smaller than 0.5, meaning that the accuracy of model $m1$ is lower compared to model $m2$, but not significantly lower. The MSVAR model has the highest accuracy compared to the other models, especially for private equity, but not significantly higher. The MSAR model has a higher accuracy than the ERS model for all equities, but not significantly higher. Both findings are in line with the calibration backtests in the sense that only the MSVAR model has correct calibration for all equities and that the MSAR model has on average higher p-values of the calibration tests compared to the ERS model.

Table 5: P-values of comparative backtest.

m1 and m2	emerging market equity	developed market equity	private equity
ERS and MSAR	0.208	0.162	0.419
ERS and MSVAR	0.239	0.283	0.115
MSAR and MSVAR	0.479	0.405	0.142

Note: For each model a moving window of 90 observations is used to forecast. The out of sample is from 2007 September to 2019 June.

5 Conclusion

In this thesis, we capture regime structures of emerging market equity, developed market equity and private equity with ERS, MSVAR and MSAR models. With these models we forecast one month ahead VaR of each equity. These forecasts are tested for correct calibration and are compared to each other in terms of accuracy. We find that the ERS and MSAR models characterise state 1 for emerging market as more volatile with a trend following effect compared to state 0. For developed market equity, state 1 also has a lower intercept compared to state 0, but with no trend following effect. For private equity state 1 has both a lower intercept and a trend following effect. There can be multiple reasons why there are trend following effects in crisis periods for emerging market equity and private equity, but not for developed market equity. One possible reason is that speculators buy stocks of less risky assets in the developed market opposed to more risky assets in emerging market and private equity in crisis periods because they speculate that stock prices will eventually increase after crisis periods which can cause this difference in trend following effect. The MSVAR model characterises state 1 similarly, except there are no trend following effects and there is a higher covariance between the error terms in state 1. The identification of regimes is important for investors to decide when to buy or sell an equity. Also taking the state dependent covariance of the error terms into account is relevant for diversification strategies of investors. The estimated correlation parameter ρ for the developed market equity and private equity is close to -0.99 and significant. This suggests leverage effects in developed market equity, and both leverage and trend following effects in private equity. With regard to the research questions, including endogeneity allows us to capture different dynamics, but it does not improve the VaR forecasts. The MSVAR model forecasts better in terms of correct calibration compared to the other models. Also the accuracy of the MSVAR model is the highest based on the comparative backtest, but not significantly better. A question arises for investors and regulators, namely what are the underlying reasons for better forecasts obtained from the MSVAR model. It could be due to the importance of capturing the covariance of the error terms, or the MSVAR model identifies recessions more precisely because of the assumption of one underlying state process for all equities, or a combination of both.

A limitation of this thesis is model specification testing. We can for example compare standard errors based on both Hessian and gradient methods, as these standard errors should be approximately equal when the model is correctly specified. Another limitation is the small size of the data set. A moving window of 90 observations is used for estimation. One may use larger moving windows to reduce estimation errors. Also the size of the out-of-sample could be larger, such that the backtests become more accurate because the limiting distributions of the test statistics only hold asymptotically. There is a trade-off since too large samples can also fail to capture possible structure changes. Furthermore, it might also be relevant to research whether the dynamics of the equity returns are different for different time frequencies, for example daily or weekly. Lastly, with respect to the VaR forecasts of the ERS model, a limited amount of different initial points for the parameters is used.

For further research, one could use macroeconomic variables to explain the conditional mean and variance of the equity returns. When the model is correctly specified, one can investigate whether endogenous regime switching is present by comparing the fit including endogeneity with the fit excluding endogeneity. The macroeconomic variables can also be implemented in the threshold for regimes transitions of the ERS model. Besides the use of macroeconomic variables, it is possible to model multiple regimes since there may be more than two regimes in reality. Lastly, one can take a Bayesian approach to take the possible randomness of the parameters into account. A multivariate version seems reasonable since there are signs of correlation between the error terms of the equity returns.

6 Appendix

First let us denote $\boldsymbol{x_{t-1}} := (1, \boldsymbol{y_{t-1}})'$ and $\Gamma_k := (\boldsymbol{\alpha}_k, H_k)$ before the maximisation.

We maximise $\log(f_k(\boldsymbol{y_t}))$ with respect to Γ_k by using the following:

$$\begin{aligned}\frac{d(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}})'\Sigma_k^{-1}(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}})}{d\Gamma_k} &= \Sigma_k(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}})\boldsymbol{x_{t-1}}' + \Sigma_k'(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}})\boldsymbol{x_{t-1}}' \\ &= -2(\Sigma_k(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}})\boldsymbol{x_{t-1}}'),\end{aligned}$$

to fill in equation (2):

$$\sum_{t=1}^{T} p_k^*(t)\frac{d\log[f_k(\boldsymbol{y_t})]}{d\Gamma_k} = \sum_{t=1}^{T} p_k^*(t)(\Sigma_k(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}})\boldsymbol{x_{t-1}}') = 0,$$

which implies that $\hat{\Gamma}_k = (\sum_{t=1}^{T} p_k^*(t)\boldsymbol{y_t}\boldsymbol{x_{t-1}}')(\sum_{t=1}^{T} p_k^*(t)\boldsymbol{x_{t-1}}\boldsymbol{x_{t-1}}')^{-1}$, where we assume that Σ_k is nonzero and the inverse of $(\sum_{t=1}^{T} p_k^*(t)\boldsymbol{x_{t-1}}\boldsymbol{x_{t-1}}')$ exists.

Now we maximise the $\log(f_k(\boldsymbol{y_t}))$ with respect to Σ_k. We use the following:

$$\frac{d\log|\Sigma_k|}{d\Sigma_k} + \frac{d(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}}')\Sigma_k^{-1}(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}})}{d\Sigma_k} = \Sigma_k^{-1} - \Sigma_k^{-1}(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}})(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}})'\Sigma_k^{-1},$$

to fill in equation (2):

$$\sum_{t=1}^{T} p_k^*(t)\frac{d\log[f_k(\boldsymbol{y_t})]}{d\Sigma_k} = \sum_{t=1}^{T} p_k^*(t)(-\frac{1}{2}(\Sigma_k^{-1} - \Sigma_k^{-1}(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}})(\boldsymbol{y_t} - \Gamma_k \boldsymbol{x_{t-1}}')\Sigma_k^{-1})) = 0,$$

which implies that $\hat{\Sigma}_k = (I_k \sum_{t=1}^{T} p_k^*(t))^{-1} \sum_{t=1}^{T} p_k^*(t)(\boldsymbol{y_t} - \hat{\Gamma}_k \boldsymbol{x_{t-1}})(\boldsymbol{y_t} - \hat{\Gamma}_k \boldsymbol{x_{t-1}})'$ where we assume that the inverse of Σ_k exists.

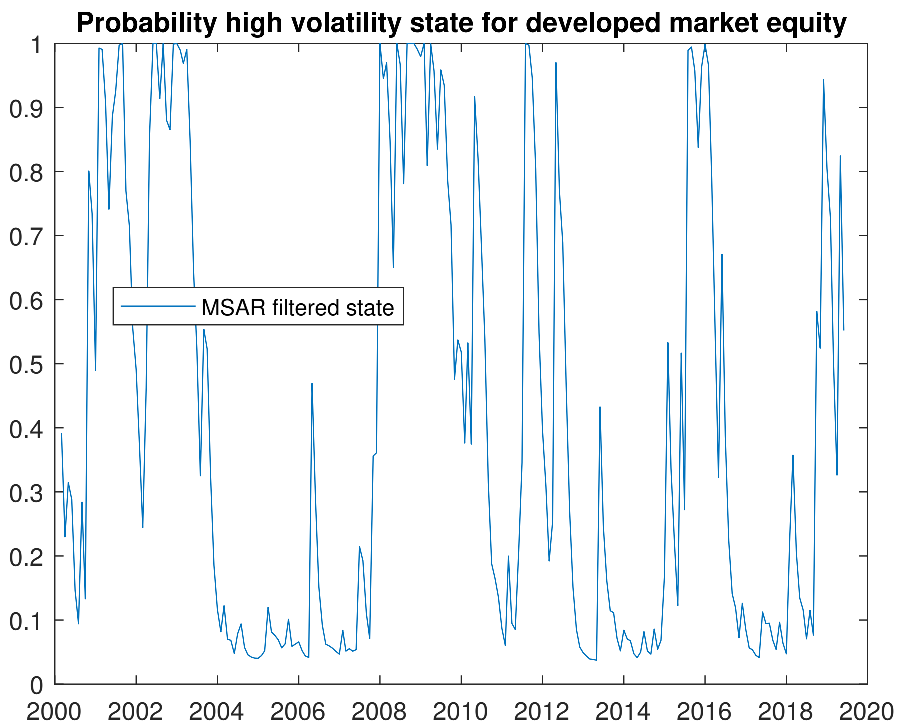

Figure 9: Filtered probabilities of being in high volatility state for developed market equity obtained from the MSAR model. Sample is from 2000 March to 2019 June.

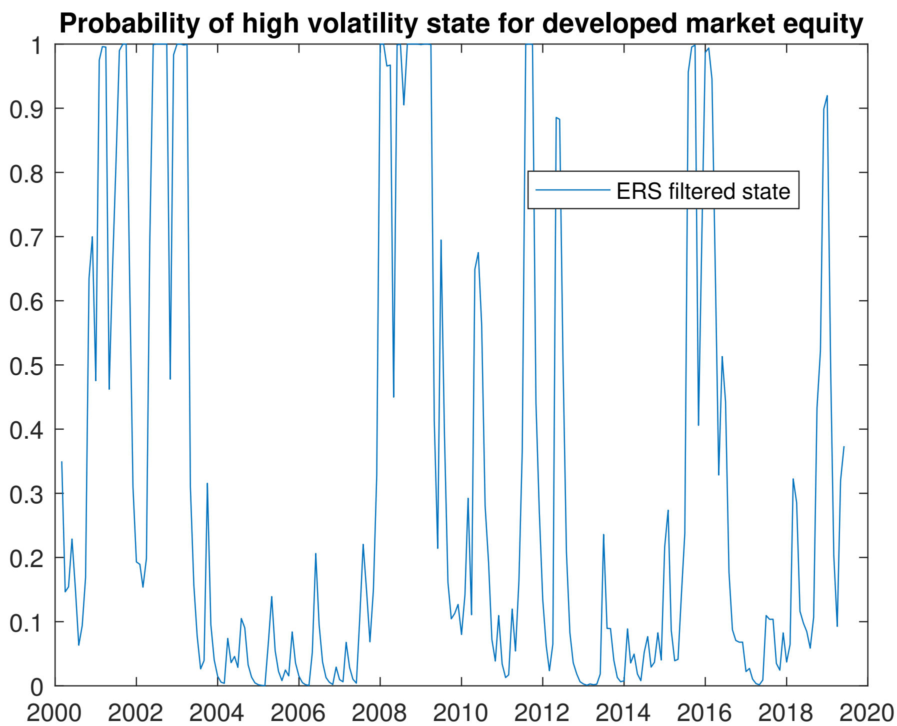

Figure 10: Filtered probabilities of being in high volatility state for developed market equity obtained from the ERS model. Sample is from 2000 March to 2019 June.

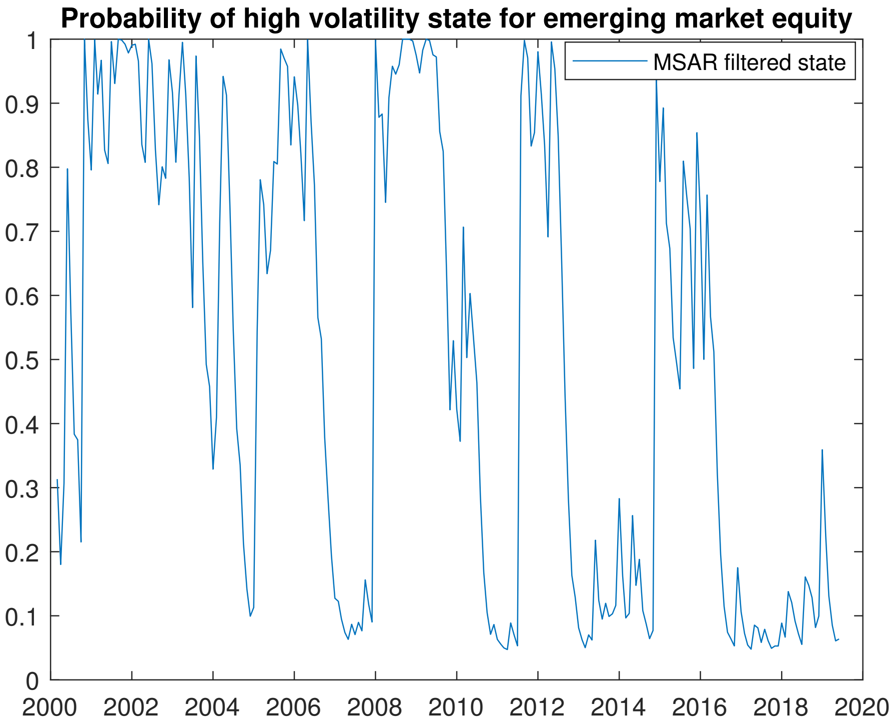

Figure 11: Filtered probabilities of being in high volatility state for emerging market equity obtained from the MSAR model. Sample is from 2000 March to 2019 June.

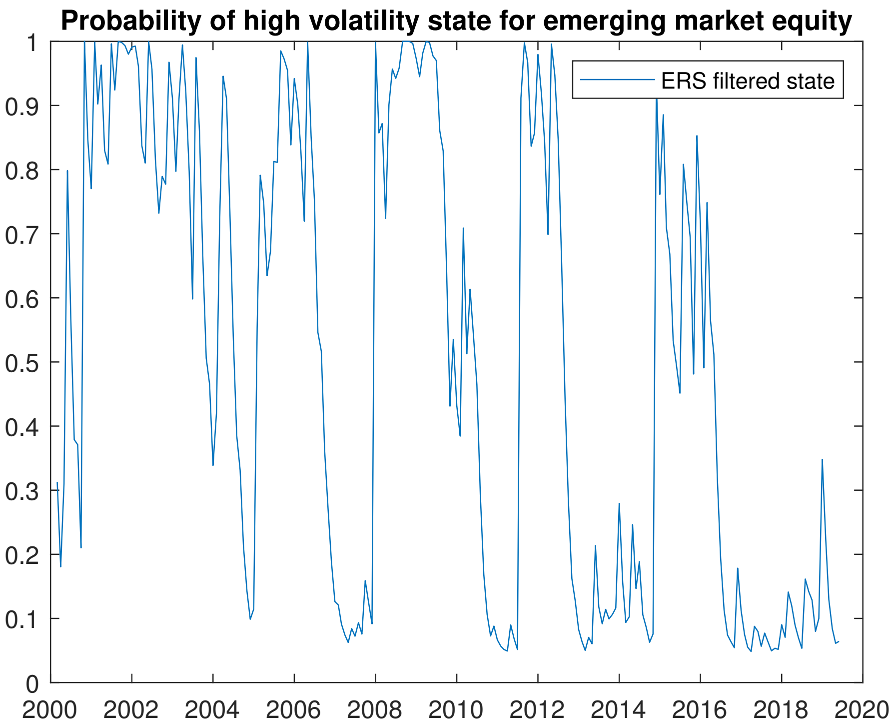

Figure 12: Filtered probabilities of being in high volatility state for emerging market equity obtained from the ERS model. Sample is from 2000 March to 2019 June.

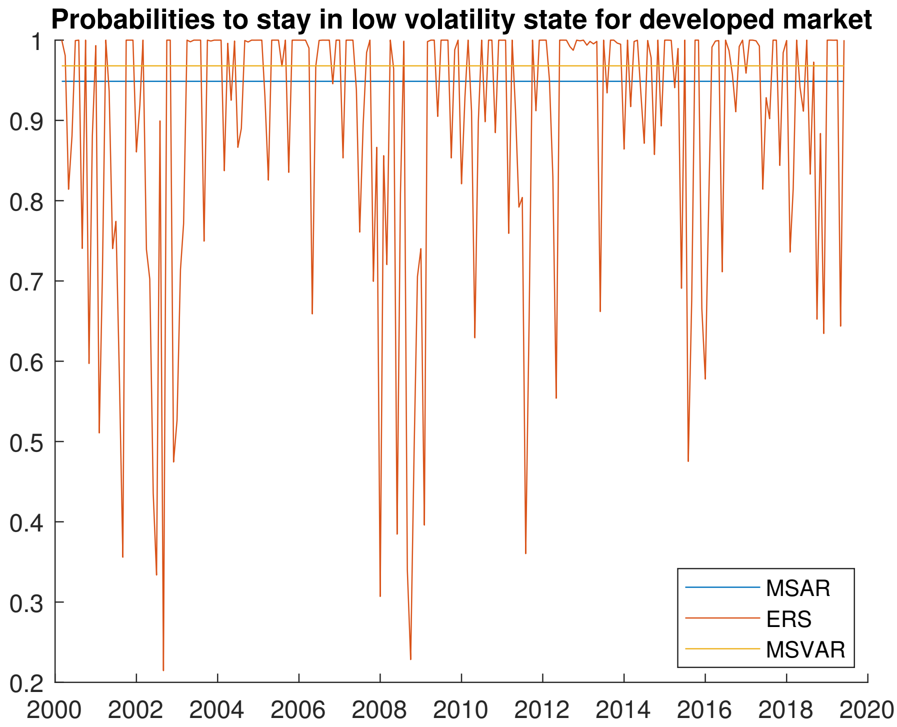

Figure 13: Probabilities to stay in low volatility state for developing market equity. Sample is from 2000 March to 2019 June.

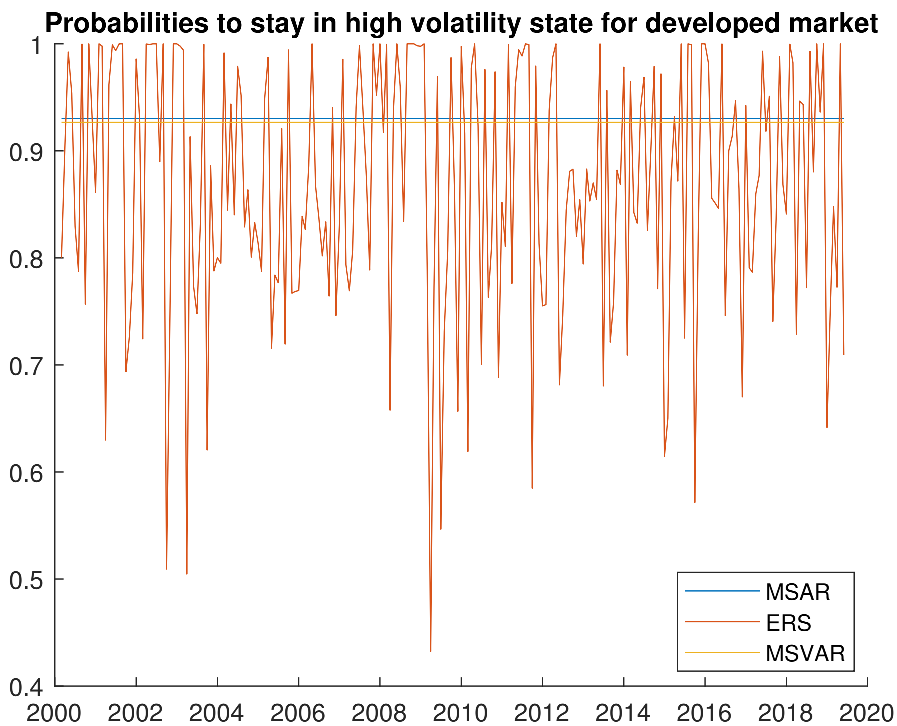

Figure 14: Probabilities to stay in high volatility state for developing market equity. Sample is from 2000 March to 2019 June.

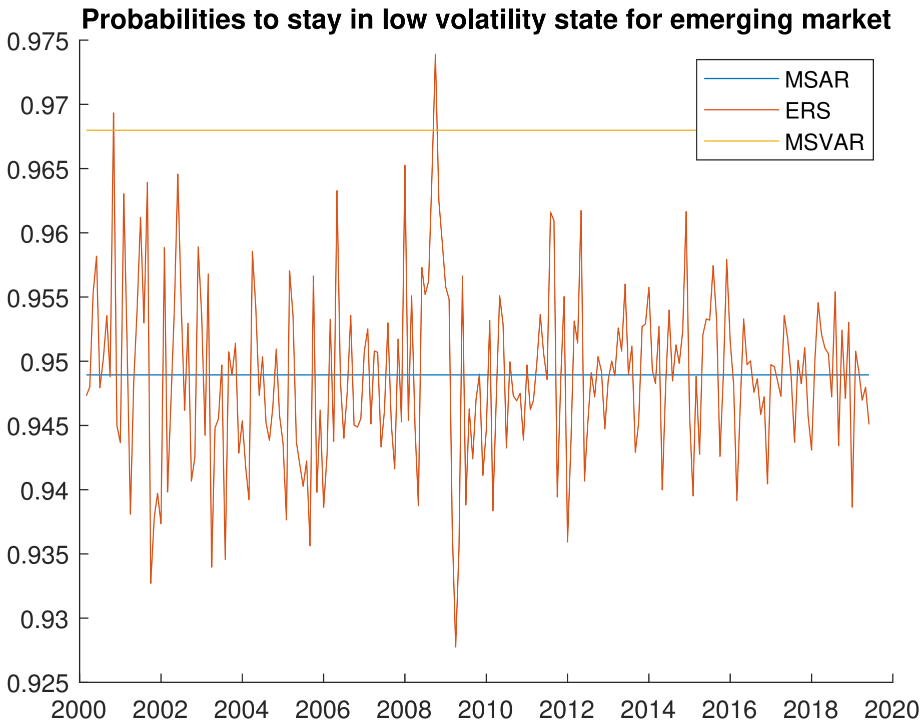

Figure 15: Probabilities to stay in low volatility state for emerging market equity. Sample is from 2000 March to 2019 June.

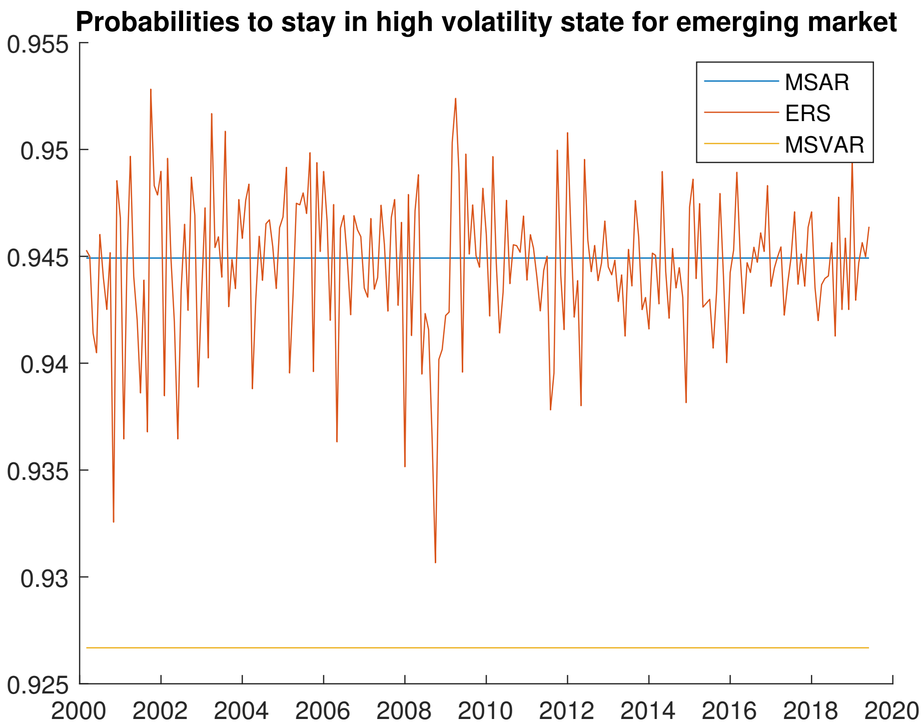

Figure 16: Probabilities to stay in high volatility state for emerging market equity. Sample is from 2000 March to 2019 June.

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