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Portfolio selection with heavy tails

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1 Abstract

Consider the problem of portfolio selection with heavy-tailed assets. According to Hyung & de Vries (2017), adding the second order expansion to the Value at Risk downside risk measure leads to more balanced portfolio's. This paper investigates whether the second order expansion is also more effective when estimating the Value at Risk. For different portfolio's consisting of stocks, bonds and commodities, we compare Value at Risk portfolio's based on second order expansion to first order portfolio's and to Extreme Risk Index portfolio's. Then we perform different violation test to measure the effectiveness of the second order expansion. We find that the second order expansion indeed gives more balanced solutions, but is not more effective when estimating the Value at Risk.

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2 Introduction

Portfolio selection is an important aspect for creating a diversified investment portfolio. The theorem of portfolio selection was first introduced by Markowitz (1952) and is nowadays still of great importance for investors. Imagine the problem of choosing between different types of assets and selecting the best portfolio weight for each asset. Financial theories such as the the mean-variance criterion of Markowitz (1952) and the capital asset pricing model (CAPM) of Sharpe (1964) assume normally distributed returns for this problem. However, the trade-off between risk and return depends on the heavily-tailed distribution of the asset returns. Fat tails are a characteristic attribute of asset returns, especially for speculative assets (Danielsson et al., 2013). The fat tail property is therefore an important feature to incorporate when measuring the risks and losses of an investment. In other words, it is an important component for selecting the most suitable portfolio.

Improvements of portfolio selection methods are always considered relevant. Various assets have different reactions on market events, as some assets react more extremely than others. According to Bondt & Thaler (1985), the stock market tends to overreact to unexpected news because of the heavy tails of the stock indices. This is why improvements of portfolio selection add to the scientific relevance and why creating a well-diversified portfolio is of great importance. Hence, using a suitable criterion measure is thus of great benefit when selecting the best portfolio weights. All types of different investors like banks, pension funds, investments companies and individual investors can benefit from a more balanced portfolio. Therefore, this research can be conducted for practical purposes.

According to Hyung & de Vries (2007), a bonds only portfolio is often selected when using the safety first criterion as a downside risk criterion. This is because the safety first criterion often incorporates Value at Risk (VaR) with only the first order Pareto term of the portfolio return distribution, which leads to corner solutions. In this paper we are going to re-create and extend the research of Hyung & de Vries (2007). They conclude that incorporating the VaR with second order tail probability in the safety first criterion, a downside risk criterion, results in a more diversified portfolio. However, we do not know the effectiveness of estimating the VaR with second order expansion. That is why we are going to measure the effectiveness of adding the second order expansion to the VaR. We aim to answer the following research question:

Will adding the second order expansion be more effective when estimating the Value at Risk?

We will also answer the following sub-questions:

1. *For the given portfolio sets consisting of a combination of different assets, does the second order expansion give a more balanced solutions?*

2. *Does the Extreme Risk Index outperform the Value at Risk portfolio's with second order expansion?*

The used data for this paper consist of the daily returns of six assets, two stock indices, two bond indices and two commodity indices. The following indices are used: the S&P 500, EURO STOXX, ICE BofA AA US Corporate Index Total Return Index Value, JPMorgan Government bond, Crude Oil and Gold. The data consist of 2487 daily observations and the data period ranges from 04-01-2011 until 30-12-2020.

We investigate a VaR method based on the second order expansion which assumes unequal tail indices and a VaR method based on the first order for which we need to assume equal tail indices. We compare the effectiveness of VaR estimates based on the second order expansion by backtesting with different violation tests. Adding to this, we will also look into another interesting optimization method, the Extreme Risk Index (ERI). The ERI is also desigined for heavy tailed assets and hence focuses on the downside risk of an investment portfolio, but unlike the VaR based on the second order expansion, the ERI assumes equal tail indices. In Mainik et al. (2015), the ERI even outperforms the minimum variance portfolio's (MV) and equally weighted portfolios (EW). Also, in contrast to the VaR based on the second order expansion, the ERI aims to exploit a form of dependence between assets. This could help lead to higher portfolio returns and makes us pose the question whether this optimization method could outperform the VaR portfolio's with second order expansion.

We start by determining portfolio sets consisting of a mix of two different assets. For each asset separately a corresponding bootstrap value and tail index will be determined which we need to compute the VaR levels for the first and second order expansion. With the VaR levels the safety first portfolio's can be determined. We can state that safety first portfolio's with second order expansion indeed lead to more balanced solutions compared to portfolio's based on VaR with first order, but portfolio's based on second order expansion don't provide higher portfolio returns. When considering portfolio sets based on related asset, it can be stated that ERI portfolio's outperform the second order VaR portfolio's, when using the ERI and VaR as a downside risk measure. After performing the unconditional coverage test and the independence test, we can state that on average adding the second order expansion isn't more effective when estimating the VaR. This is because the VaR estimates based on second expansion are too conservative. Moreover, for some portfolio's no violations are found, which could also indicate that the VaR estimates are too high. However, the used violation tests are not defined in case there are no violations. Therefore, no proper conclusion can be stated from these results.

This paper is organised in the following manner: in Section 3 and Section 4 we will provide information regarding the literature and theory of the research problem and the used methods. In Section 5 the used data will be shown. In Section 6 we present the portfolio sets, the safety first criterion, the methods for estimating the VaR with the first and second order expansion, the ERI and the used violation test for backtesting the VaR. Section 7 will present the results of our used methods and in Section 8 the conclusion is stated.

3 Literature

The mean-variance criterion was first proposed by Markowitz (1952) to tackle the portfolio selection problem. This optimization method considers selecting portfolio's based on the mean and variance of the returns. The mean and variance are then used as a measure for selecting the most suitable portfolio for each individual investor based on a trade-off between risk and return. Since periods of large losses occur more often than periods of large returns, risk-managers nowadays apply downside risk measures to define the risks of the portfolio assets. Therefore, a good alternative for risk-avers investors is selecting portfolio's based on the safety first criterion (Roy, 1952; Arzac & Bawa, 1977). VaR is used as a downside risk measure for the safety first criterion in Jansen et al. (2000). Using the property of heavy tailed assets and the semi-parametric method, they compute the optimal portfolio based on VaR consisting of stocks and bonds. VaR is often used for empirical research, when there are heavy tailed risk factors (Glasserman et al., 2002). Yet, only a small part of the literature works out the theoretical aspects of the VaR or the consequences of the risk measure, that is why Gourieroux et al. (2000) present an analysis of the convexity and the sensitivity of VaR.

When portfolio optimization is done based on the tail properties of the distribution of the asset returns, it's often the case that assets with high tail coefficients will be chosen, which leads to unbalanced portfolio's (Straetmans, 1998; Jansen et al. 2000; Hartmann, Straetmans & de Vries, 2004). As a solution to this problem, Hyung & de Vries (2007) incorporate the second order expansion into the VaR. An elaboration of the validity of the second order is given by Geluk & de Haan (1987). They prove that when considering a convolution of two variables, the variable with the fattest tail will dominate, making the application of the second order to the VaR possible. The paper by Hsu et al. (2012) analyses the effectiveness of the VaR based on copula-extreme-value semi-parametric approaches. They examine the effectiveness based on backtesting analysis. Uylangco & Li (2016) compare VaR estimates based on four different analytical techniques, namely using a parametric approach, a historical approach, Monte Carlo simulation and an autoregressive conditional moving average-generalized autoregressive conditional heteroskedasticity (ARMA-GARCH) models.

Comparing different downside risk measures and different optimization methods can be useful when selecting the optimal portfolio. Mainik et al. (2015) perform the ERI to minimize the likelihood of large drawdowns. They prove that, given heavy tails, the ERI method beats the equally weighted portfolio and the minimum variance portfolio. The ERI method is often combined with a Multivariate Regular variation (MRV) framework. To compare the affectibility of portfolio losses to extreme events, Mainik & Rüschendorf (2010) use MRV to model the losses of the assets. They use an extreme index which consist of equal tail indices, α, for all heavy tailed assets and the probability measure Ψ, which can also be referred to as the spectral measure. Unlike the second order VaR, which only considers the dominating terms of the marginal distribution, the MRV assumes the joint distribution of the assets. An MRV model is also used by Mainik & Embrechts (2012) to analyze the consequences of asymptotic diversification for portfolio's with heavy tailed assets and for investigating the sub-additivity and the convexity of VaR.

4 Theory

4.1 Extreme Value Theory and tail probabilities

Before applying the safety first criterion, we first need to present the tail probabilities considering the heavily-tailed distribution of the asset returns. Instead of using an exponential decay of the survival function, such as for the normal distribution, the tails of a heavily-tailed distribution are managed with a power rate. The tail probabilities are established via the extreme value theory (EVT).

The first order tail probability is as follows, (Hyung & de Vries, 2007)

$$P\{X_i > s\} = A_i s^{-\alpha_i} + o(s^{-\alpha_i}), \tag{1}$$

where the relationship holds as s $\rightarrow \infty$, α and A > 0 for i = 1,..,n. We assume that the returns are i.i.d. and the tails are regularly varying at infinity. Because we consider minimizing the downside risk of the portfolio, we look at the left tail of the distribution. Therefore, X_i represents the loss of returns of asset i and s represents a large loss threshold. The second term in the first order tail probability, $o(s^{-\alpha_i})$, represents the limit notation: $lim_{s\rightarrow 0}$\{P\{X_i > s\} - As$^{-\alpha_i}$\} $\cdot$ s^{α_i} = 0 (Gay, 2005; Janson, 2011). More generally, g(s) = 0(f(s)), where $\frac{g(s)}{f(s)} \rightarrow 0$.

The second order tail probability or one of the two non-trivial expansions derived from the limit function, used in de Haan & Stadtmüller (1996), is as follows (Hyung & de Vries, 2007),

$$P\{X_i > s\} = A_i s^{-\alpha_i}[1 + B_i s^{-\beta_i} + o(s^{-\beta_i})], \tag{2}$$

where we assume s $\rightarrow \infty$ and α, β and A > 0 for i = 1,..,n. B $\in R$. The term $o(s^{-\beta_i})$, follows the same limiting notation as $o(s^{-\alpha_i})$ (Gay, 2005; Janson, 2011). The first order tail index is the scale parameter α, and β is the second order tail index (Dacarogna et al., 2001). The expansion can be applied to all heavily-tailed distributions. The other non-trivial expansion used in de Haan & Stadtmüller (1996) will not be elaborated in this paper.

We investigate the second order expansions of different types of assets: stocks, bonds and commodities. To create a balanced investment portfolio, we take the weighted sum of the assets. That is why we need to use the convolution of two assets, for example X1 + X2. We use two cases for each convolution: equal tail indices with $\alpha_1 = \alpha_2$ and unequal tail indices $\alpha_1 \neq \alpha_2$. For the equal tail indices, the tail index with the fattest tail will be chosen, because according to Geluk & de Haan (1987) the fattest tail overrules. For the unequal tail indices we employ Theorem 1 from Hyung & de Vries (2007), which states different cases for the convolution with second order terms. The following case from Theorem 1 is relevant: if α_2 - α_1 < min(β_1,1), then P\{$X_1 + X_2$ > s\} = $A_1 s^{-\alpha_1}$ + $A_2 s^{-\alpha_2}$ + o($s^{-\alpha_2}$).

4.2 Convexity of Value at Risk

By taking into account the second order term of the tail probability, we expect a more diversified portfolio. The risk measurement needs to be convex to further establish this (Gourieroux et al., 2000). We use the function of the one-day VaR at 100 x (1-q)% (Andersen et al., 2006). VaR is a risk measure that represents the quantile, $q*$, of the conditional return distribution, where the cumulative distribution function is conditioned on I_t (Andersen et al., 2006):

$$P[r_{t+1} \leq VaR_t(1-q,1)|I_t] = F_{t+1|t}(\text{VaR}_t(1-q,1)) = q* . \tag{3}$$

We consider two different types of assets X_i, i= 1,2 with corresponding weights w per portfolio set. The VaR for two assets is as follows: $\text{P}\{\omega X1 + (1-\omega)X2 > \text{VaR}(\omega, q)\} = q$, where q represents the loss probability threshold (Hyung & de Vries, 2007).

To prove convexity of the VaR, Gourieroux et al. (2000) present a general notation of the first and second derivatives. Nevertheless, we first need to obtain the derivatives of the tail probabilities to derive the first and second-order derivatives of a heavily-tailed distribution. To determine the convexity of the tail probabilities, we use Proposition 2 from Hyung & de Vries (2007). By deriving the hessian of the second order tail probabilities in Proposition 2, it can be shown that the convolution of the second order tail probability for two assets is convex.

Our aim is to minimize the VaR so we can determine the optimal weights for the VaR risk measure. This should results into a minimization of the losses of the portfolio assets and if this risk measure is also convex it ensures more diversified investment portfolio's. That is why, knowing the convexity of the tail probabilities, we need to determine the convexity of VaR using Proposition 3 from Hyung & de Vries (2007), only now we consider the asymptotic inversion from the VaR derived from De Bruijn's theory:

$$\text{VaR}(\omega, q) = \omega Y_1^{\frac{1}{\alpha_1}} q^{-\frac{1}{\alpha_1}} [1 + \frac{(1-\omega)^{\alpha_2}}{\omega^{\alpha_2}} \frac{Y_2}{\alpha_1 Y1^{\alpha_2/\alpha_1}} q^{\frac{\alpha_2-\alpha_1}{\alpha_1}}]. \tag{4}$$

Again, by deriving the hessian of the VaR based on the second order expansion, the convexity of the function of the VaR for two assets can be shown. The proof for convexity can be found in Hyung & de Vries (2007). Because the VaR method is convex, we can conclude that the VaR criterion ensures more diversification in the investment portfolio.

5 Data

For this research we use the daily logarithmic returns of six assets: two stock indices, two bond indices and two commodity indices. For the stock indices we use the Standard & Poor's 500 (S&P 500) and the EURO STOXX. Both indices consist of a mix of different types of stocks, making them a good representative for the stock market. For the bond indices we use the ICE BofA AA US Corporate Index Total Return Index Value (Corporate bond) and JPMorgan Government Bond

Index (Government bond). We cover diverse bonds so we get an improved insight in bonds overall. For example, it's known that corporate bonds tend to have more risk than government bonds. For the commodity indices we look at the Gold index and the Crude Oil index, because gold and oil are among one of the oldest and most traded commodities. The daily returns of the six indices are obtained from Yahoo Finance and the Federal Reserve Economic Data (FRED). The data is filtered in such a way that we have an equal amount of observations for each asset, corresponding to the same data. The data consists of 2487 daily observations and the data period ranges from 04-01-2011 until 30-12-2020. The chosen data period consists of diverse periods either with or without a financial crisis.

For the statistical properties we are going to transform the daily returns into the logarithmic daily returns. The returns r_{it} of index i at time t as follows:

$$r_{it} = 100 \cdot \log\left(\frac{P_t}{P_{t-1}}\right),$$

where P_t represents the price of the asset at time t and log the logarithmic function. In Table 1, a summary of the statistical properties of the logarithmic daily returns of the six assets is presented. The table shows that the means are close to zero, only the mean of Crude Oil is slightly negative. The Government bond has the lowest standard deviation, which is plausible considering the low default risk of the government. The skewness is negative for all asset returns except for Crude Oil. The kurtosis of all assets returns are much higher than the expected kurtosis for a normal distribution, which suggests a heavily-tailed distribution for all assets. Lastly, the Jarque-Bera (JB) test statistics are large for every index. We can perform a normality test with the JB test statistics. With p-values equal to 0.000, we can reject the null hypothesis of normality. The histograms in Figure 1 in Appendix A confirm the characteristics of non-normality and heavy tails for the asset returns. It's important for our research to have non-normally distributed assets because the used tail probabilities, displayed in equation (1) and (2), assume heavily-tailed distributions.

Table 1: Summary of the statistical properties of the logarithmic daily returns of the six assets for the full sample period 04-01-2011 until 30-12-2020.

	Mean	Std.dev.	Skewness	Kurtosis	JB
S&P 500	0.0434	1.1112	-0.9394	20.5661	32504***
EURO STOXX	0.0133	1.3359	-0.6951	11.2325	7294***
Gold	0.0127	1.0356	-0.6314	10.1897	5533***
Crude Oil	-0.0247	2.7130	-3.2886	32.5890	90688***
Corporate*	0.0185	0.2787	-1.2127	17.4946	22345***
Government*	0.0014	0.2349	-0.2999	5.4277	647***

Notes: The mean is presented in percentages. JB represents the outcome of the Jarque-Bera test.
* Corporate stands for Corporate bond and Government stands for Government bond.
*** 1 % significance.

6 Methodology

6.1 Portfolio sets

Before the VaR levels for the safety first portfolio's can be determined we choose the combination of assets in a portfolio. We want to investigate the effect of the second order expansion by using portfolio's which contain diverse assets. That is why, we create three portfolio sets, each portfolio set consists of two assets which are low correlated. The correlations between the six asses can be seen in Table 2. The following portfolio sets are specified: Portfolio set 1: S&P 500 and Gold, Portfolio set 2: EURO STOXX and Corporate bond and Portfolio set 3: Crude Oil and Government Bond.

Table 2: Correlations of the six assets during the full sample period 04-01-2011 until 30-12-2020.

	S&P 500	EURO STOXX	Crude Oil	Gold	Government*	Corporate*
S&P 500	1.0000	-0.0325	0.0276	-0.0269	0.0560	0.0484
EURO STOXX	-0.0325	1.0000	-0.0229	-0.0269	-0.3269	-0.1788
Crude Oil	0.0276	-0.0229	1.0000	-0.0169	-0.0163	0.0014
Gold	-0.0269	-0.0269	-0.0169	1.0000	0.2389	0.269
Government*	0.0560	-0.0327	-0.0163	0.2389	1.0000	0.8095
Corporate*	0.0484	-0.1788	0.0014	0.2559	0.8095	1.0000

Note: *Corporate stands for Corporate bond and Government stands for Government bond.

6.2 Safety first criterion

Portfolio selection based on the safety first criterion deals with the expected returns and risks of financial assets. Adding risk as a factor was originally published by Roy (1952) and later elaborated by Arzac & Bawa (1977). For the elaborations of the safety-first criterion we will follow Arzac & Bawa (1977), just like Jansen et al. (2000) and Hyung & de Vries (2007). The safety first problem maximizes the expected return μ for the corresponding value of π. $\mu = \mathrm{E}[\Sigma_i \omega_i P_{it}] + \mathrm{a}$, where ω_i is the weight corresponding to the risky asset i in the portfolio and P_{it} corresponds to the market prices of asset i at time t. Furthermore, π corresponds to the value of the probability of negative returns:

$$\begin{cases} \pi = 1 & \text{if} \quad p = P(\omega_i P_{it} + a * r \leq s) \leq \delta, \\ \pi = 1 - p & \text{if} \quad p = P(\omega_i P_{it} + a * r \leq s) > \delta, \end{cases} \tag{5}$$

where s represents the disaster level of the net worth and δ represents the critical value of the probability of this disaster. r is the rate of the risk-free return. If a is positive, it represents the amount lend by the investor, and if a is negative, it represents the amount borrowed by the investor.

We can now state the safety first portfolio problem as follows,

$$max(\pi, \mu) \tag{6}$$

$$\sum_i \omega_i P_{it} + a = W_t \qquad \forall i, t \geq 0, \tag{7}$$

where the net worth of the investor at time t corresponds to W_t. In the objective function, the trade off between risk and returns is presented. For the safety first portfolio problem, Jansen et al. (2000) derive the maximum risk-premium to return opportunity loss ratio with corresponding probability δ:

$$max \frac{(\bar{R} - r)}{(r - q_\delta(R))}, \tag{8}$$

which is used to determine the safety first portfolio's. We can calculate the gross returns R by R = $\frac{\Sigma \omega_i P_{it+1}}{\Sigma \omega_i P_{it}}$ and $\bar{R}$ is the expected value of R. q_δ(R) represents the quantile of the VaR risk measure.

6.3 Value at Risk with first order and second order expansion

After providing the theory of the tail probabilities, we can apply the first order and second order expansion to the safety first criterion. The VaR is a quantile that represents the maximum loss that will not be surpassed over a given period with a probability level of δ%. We will look at three different failure probabilities related to the sample size: δ = 1/n, δ = 0.75/n and δ = 0.5/n, in words we could describe δ = 1/n, as one failure out of 2487 observations. Following Jansen et al. (2000), we estimate the quantiles of the VaR based on the first order, $q_{1,\delta}$, for each portfolio with pre-determined weights ω, as follows,

$$\hat{q}_{1,t} = X_{(m_i)} \frac{m_i}{n\delta}^{\frac{1}{\hat{\alpha}_i}}, \tag{9}$$

where n represents the sample size, m corresponds to the optimal order statistic value, which will be estimated by the bootstrap method of Hall (1990). An explanation of the bootstrap method can be found in Appendix B. The first order tail index $\hat{\alpha}$ is calculated using the Hill (1975) moment estimator:

$$\frac{1}{\hat{\alpha}} = \frac{1}{m} \sum_{i=1}^{m} [Log(X_{(n+1-i)} - Log X_{n-m}], \tag{10}$$

where $X_1 \leq X_2 \leq .. \leq X_m \leq$ ..X$_n$ corresponds to the order statistics of the distribution function with samples X_1, X_2... X_n. For the first order tail probability we consider equal tail indices for portfolio's that combine two assets (Jansen et al., 2000).

We now consider the VaR with second order expansion for each portfolio for predetermined weights. Using Proposition 3 and note 2 of Hyung & de Vries (2007), we calculate the quantiles of the VaR with second order expansion, $q_{2,\delta}$, for each portfolio with pre-determined weights ω, with the following equations:

$$\omega^{\alpha_1} Y_1 q_{2,\delta}^{-\alpha_1} + (1-\omega)^{\alpha_2} Y_2 q_{2,\delta}^{-\alpha_2} \approx \delta, \tag{11}$$

$$Y_i = \frac{m_i}{n} X_{m_i}^{\alpha_i}.$$

The tail index is again estimated with the Hill (1975) moment estimator. However, for the second order expansion we consider different tail indices for each asset. To confirm that the tail probabilities satisfy the first case of theorem 1, an estimate of the second order tail indices can be derived from the following formula (Hyung & de Vries, 2007):

$$\frac{\hat{\beta}}{\alpha} = \frac{ln(m)}{2ln(n) - 2ln(m)}, \tag{12}$$

6.4 Extreme Risk Index

In comparison to the VaR with second order expansion, the ERI assumes equal tail indices and exploits the dependence between two assets, making this optimization method an interesting addition to our research. We follow the ERI optimization method, developed by Mainik et al. (2015), which minimizes the probability of large losses in an investment portfolio using the EVT. By minimizing the ERI, Mainik et al. (2015) estimate the effect of the extreme index returns on the heavy tails of the portfolio returns. Because we investigate the downside risk of the portfolio assets, we look at the logarithmic losses of portfolio asset i at time t:

$$X_i(t) = -log \cdot \left(\frac{P_{it}}{P_{i,t-1}}\right) = log(P_{i,t-1}) - log(P_{it}). \tag{13}$$

To interpret the random vector X_{it} of the ERI, Mainik et al. (2015) use the MRV. The MRV considers the joint distribution of the polar coordinates of the logarithmic losses. The polar coordinates are calculates as follows,

$$(R_t, Z_t) = \left(||X_t||_1, ||X_t||_1^{-1} X_t\right). \tag{14}$$

The polar coordinate R_t represents the length and the coordinate Z_t the angle of logarithmic losses, where $R_t = ||X_t||_1 = \sum_{i=1}^{m} |X_{it}|$. We now put the radius parts R_t in descending order: $R_{1,t} \geq ... \geq R_{n,t}$ and link the radius to the corresponding angular part.

By incorporating the radial components R_t, we estimate the first order tail index $\hat{\alpha}$ with the Hill (1975) estimator:

$$\hat{\alpha} = \frac{k}{\sum_{i=1}^{k} log(\frac{R_i t}{R_{i+1,t}})}. \tag{15}$$

Following Mainik et al. (2015), we use 10% of total number of returns which is represented by k.

We aim to estimate the optimal weights for each asset that minimize the estimator γ_w of the ERI problem:

$$\gamma_{\hat{\omega *},t} = min_{\omega,t} \gamma_{\hat{\omega_i},t}, \tag{16}$$

where

$$\gamma_{\hat{\omega_i},t} = \frac{1}{k} \sum_{i=1}^{k} \max(0, \omega^T Z(j_{i,t}))^{\hat{\alpha}}. \tag{17}$$

J_{it} represents the sample index of the ordered statistic R_i, t, where $R_i, t = R(j_{i,t})$.
We want to compare the ERI portfolio's to VaR portfolio's with second order expansion, therefore we calculate the optimal ERI portfolio given the estimated optimal weights and the optimal portfolio of the VaR based on equation (4) for an in-sample period. The two methods will be compared using two downside risk measures, again the VaR and the ERI for an out-of-sample period. The in-sample period ranges from 04-01-2011 until 30-12-2017 and the out-of-sample period ranges from 02-01-2018 until 30-12-2020.

6.5 Backtesting

To evaluate the quality of the VaR estimates we look at two aspects:

1. Correct unconditional coverage: the amount of returns that exceed the VaR must be the same as the nominal coverage probability q.

2. Independence: violations of the VaR should appear randomly and not in bundles.

A VaR estimate satisfies both aspects if it has correct conditional coverage. By performing violation tests, we can determine whether adding the VaR estimates with second order expansion is more effective compared to the VaR estimates without second order expansion. For the backtesting procedure the same in-sample and out-of-sample periods are used as for comparing the ERI to the VaR based on the second order expansion. We follow the methods and tests for the conditional forecast evaluation from Christoffersen (1998) to perform the violation tests.

The backtesting procedure starts by computing the indicator variable, I_{t+1}, which represents the hit rate for a violation of the VaR:

$$I_{t+1} = \begin{cases} 1 & \text{if} \quad r_{t+1} < VaR_t(1-q,1), \\ 0 & \text{if} \quad \text{r}_{t+1} > VaR_t(1-q,1). \end{cases} \tag{18}$$

6.5.1 Unconditional coverage test

A coverage test examines whether the number of exceedances contrast with the quantile of loss of the VaR. For the unconditional coverage test, we have the following null hypothesis:

$$H_0 : P[I_{t+1} = 1] = E[I_{t+1}] = q, \tag{19}$$

which means that the amount of returns that exceed the VaR must be the same as the nominal probability q. Using coverage probability p = P$[I_{t+1} = 1]$, we set up the following likelihood functions for the null hypotheses and the alternatives hypothesis:

$$\begin{aligned} H_0 : p = q \qquad & \mathcal{L}(q; I_T, I_{T-1}, .., I_1) = (1-q)^{T_0} q^{T_1}, \\ H_a : p = \pi \qquad & \mathcal{L}(\pi; I_T, I_{T-1}, .., I_1) = (1-\pi)^{T_0} \pi^{T_1}. \end{aligned} \tag{20}$$

The maximum likelihood estimate (MLE) for the alternative hypothesis variable $\hat{\pi}$ is defined as:

$$\hat{\pi} = \hat{P}[I_{t+1} = 1] = \frac{T_1}{T_0 + T_1}. \tag{21}$$

MLE is the VaR violation probability where T_1 represents the number of VaR violations and T_0 the number of non-violations of the VaR estimates.

We test the null hypothesis for correct unconditional coverage with the Likelihood Ratio test (LR):

$$LR_{uc} = -2 \cdot log(\frac{\mathcal{L}(q; I_T, I_{T-1}, .., I_1)}{\mathcal{L}(\hat{\pi}; I_T, I_{T-1}, .., I_1)}) \sim \chi^2(1) \tag{22}$$

in which $\mathcal{L}$ represents the likelihood function and $\hat{\pi}$ denotes the MLE of the nominal probability q.

6.5.2 Independence test

An independence test examines if the results between two consecutive periods are independent from each other. A VaR violation should be independent of the past. For the independence test the following null hypothesis is stated:

$$H_0 : P[I_{t+1} = 1 | I_t] = P[I_{t+1} = 1] \quad \forall t, \tag{23}$$

where we state that the conditional probability of violation is equal to the unconditional probability, while we condition on the indicator function. This is because we need to use the first-order Markov chain for the alternative hypotheses .

In case the indication function I_{t+1} is a first-order Markov chain, we set up the following transition probability matrix:

$$\Pi_1 = \begin{pmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{pmatrix}. \tag{24}$$

The transitions can be constructed by $\pi_{ij} = P[I_{t+1} = j|I_t = i]$. The likelihood function for the alternative hypotheses is set up as follows:

$$\mathcal{L}(\Pi_1; I_T, I_{T-1}, ...I_1) = (1-\pi_{01})^{T_{00}}(\pi_{01})^{T_{01}}(1-\pi_{11})^{T_{10}}(\pi_{11})^{T_{11}}. \tag{25}$$

T_{ij} represents the amount of observations, where I_{t+1} = j and I_t = i. For example: T_{01} is the amount of observations for which there were no violations in the first period, but violations in the next period. The MLE of Π_1 is again:

$$\hat{\Pi_1} = \begin{pmatrix} \frac{T_{00}}{T_{00}+T_{01}} & \frac{T_{01}}{T_{00}+T_{01}} \\ \frac{T_{10}}{T_{10}+T_{11}} & \frac{T_{11}}{T_{10}+T_{11}} \end{pmatrix}. \tag{26}$$

The parameters of the MLE are components of the different transition states. With the null hypotheses of independence $\pi_{01} = \pi_{11} = \pi_{02}$ and the likelihood function:

$$\mathcal{L}(\Pi_2; I_T, I_{T-1}, ...I_1) = (1-\pi_2)^{T_{00}+T_{10}}(\pi_2)^{T_{01}+T_{11}}. \tag{27}$$

We can set up the MLE of π_2:

$$\hat{\pi_2} = \frac{T_{01}+T_{11}}{T_{00}+T_{10}+T_{01}+T_{11}.} \tag{28}$$

Lastly, we can test the independence with the LR test:

$$LR_{indep} = -2 \cdot log(\frac{\mathcal{L}(\hat{\pi_2}; I_T, I_{T-1}, .., I_1)}{\mathcal{L}(\hat{\pi_1}; I_T, I_{T-1}, .., I_1)}) \sim \chi^2(1). \tag{29}$$

7 Results

7.1 Safety first portfolio's

We use the established portfolio sets to determine the bootstrap values and the first order tail indices for each asset separately to investigate whether the second order expansion gives more balanced solutions. The bootstrap values are computed in R, using packages: tea and tidyverse. Table 3 shows the values for the bootstrap and the tail indices. Notable are the values of the assets Crude Oil and Government bond. Crude Oil has a high bootstrap value and a low first order tail index, compared to the other assets. Government bond has on average a slightly higher tail index and an extremely low bootstrap value. The explanation for this is logical, considering the safer image of Government bonds and the extremely volatile sample period for Crude Oil. It can also be seen that the difference between tail indices within a market, so between stocks, bonds or commodities, can differ substantially.

Table 3: Bootstrap values and tail indices corresponding to the six assets during the full sample period 04-01-2011 until 30-12-2020.

	α	β	m	n	X_m
S&P	2.6911	1.9434	101	2487	0.0163
Gold	3.7749	1.3754	27	2487	0.0269
EURO STOXX	4.0763	1.1198	16	2487	0.0398
Corporate*	4.0803	2.0843	52	2487	0.0054
Crude Oil	1.9435	2.7512	319	2486	0.0207
Government*	6.3929	0.0000	1	2487	0.0106

Notes: α represents the first order tail index, β the second order tail index, n is the number of observations and m is the optimal order statistic value.
* Corporate stands for Corporate bond and Government stands for Government bond.

We calculate the VaR levels corresponding to the first order and second order expansion for the three portfolio sets. Each set consists of different portfolio's corresponding to different weights of two assets. The weights for the 11 different portfolio alter with 10%. The values for X_m can be found in Table 3. For the first order, a portfolio that consists of one asset uses its own corresponding tail index, whereas a portfolio that consists of a mix between two assets uses equal tail indices. For the second order expansion we use unequal tail indices. The VaR levels are computed for three different failure probabilities δ. Table 4 shows the results for $\delta = 1/\text{n}$. For the first VaR level of -0.0904 this means that 1 out of 2487 days, approximately 9% of your investment will be lost. In Portfolio set 3 it's very noticeable for both the first order and the second order expansion that a portfolio that invests more in Crude Oil can have higher losses compared to investing in the more safer asset Government bond. The VaR levels for the different portfolio's in Portfolio set 2 also state similarly that a portfolio that consists mostly of bonds, in this case Corporate bonds, leads to lower losses than a portfolio that consists of other assets, in particular the EURO STOXX asset. For Portfolio set 1, the VaR levels for the different portfolio's are more in the same range. The results for the other probabilities can be found in the Appendix C.

Table 4: Value at Risk levels of the first order and second order expansion corresponding to the pre-determined probability for the full sample period 04-01-2011 until 30-12-2020.

$\delta = 1/2487$	**First order**			**Second order**		
Weights	Portfolio set 1	Portfolio set 2	Portfolio set 3	Portfolio set 1	Portfolio set 2	Portfolio set 3
100% Asset 1	-0.0904	-0.0786	-0.4023	-0.0904	-0.0786	-0.4023
90% Asset 1	-0.0559	-0.0721	-0.0470	-0.0813	-0.0707	-0.3620
80% Asset 1	-0.0566	-0.0657	-0.0429	-0.0723	-0.0628	-0.3218
70% Asset 1	-0.0573	-0.0592	-0.0389	-0.0635	-0.0550	-0.2816
60% Asset 1	-0.0580	-0.0528	-0.0349	-0.0552	-0.0471	-0.2414
50% Asset 1	-0.0586	-0.0464	-0.0308	-0.0487	-0.0393	-0.2011
40% Asset 1	-0.0593	-0.0400	-0.0268	-0.0456	-0.0315	-0.1609
30% Asset 1	-0.0600	-0.0336	-0.0227	-0.0466	-0.0237	-0.1207
20% Asset 1	-0.0607	-0.0272	-0.0187	-0.0505	-0.0167	-0.0805
10% Asset 1	-0.0614	-0.0207	-0.0146	-0.0559	-0.0133	-0.0402
0% Asset 1	-0.0620	-0.0143	-0.0106	-0.0620	-0.0143	-0.0106

Notes: Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).

Now we compute the safety first portfolio's with downside risk for different values of δ using the estimated VaR levels for the first and second order expansion. The safety first portfolio's computed for different risk-free rates can be seen in Table 5 and Table 6. Following Jansen et al. (2000), we choose two risk-free rates, a risk-free rate equal to 1 so no payments for the risk-free asset and a risk-free rate approximately equal to 1.0015 with a yearly 3.7 % interest rate. We hereby assume approximately 250 trading days in a year. When the risk-free rate equal to 1, it can be seen that for Portfolio set 1, the best portfolio with a second order expansion gives lower portfolio returns then the best portfolio with a first order. Yet, for the second order expansion a more balanced portfolio of a mix between the S&P and Gold (50/50) gives the best portfolio returns. For Portfolio set 2, we can conclude that both first order and second order expansion suggest to invest the majority in the Corporate bond stock, where the first order suggest that a corner solution gives the best portfolio and the second order expansion suggest a slightly more balanced portfolio. Portfolio set 3 shows a contradictory result, because the two methods suggest using almost the opposite mixing strategy for the two assets Crude Oil and Government bonds. While the first order suggest a more risky portfolio is better, containing most investments in Crude Oil, with higher returns. The second order expansion suggests using the portfolio with an asset with a much higher tail index and less returns, resulting in a portfolio where we invest mostly in the Government bond. For a risk-free rate equal to 3.7% interest per year, results show similar solutions as the portfolio's with zero net returns, but for a portfolio that consists of only Government bonds the results show negative portfolio returns, which means that for this interest rate it is not profitable to use the safety first method for given portfolio sets.

Overall it, can be stated that for the given portfolio sets and considering different failure probabilities that, on average portfolio's containing second order expansion indeed give more balanced solutions than portfolio's without second order expansion, but first order portfolio's do give slightly higher returns. However, for the used portfolio sets the difference between portfolio's with first or second are very small, except for Portfolio set 3.

Table 5: First order Value at Risk portfolio's for the full sample period 04-01-2011 until 30-12-2020.

$\delta = 1/2487$	r = 1			r = 1.0015		
Weights	Portfolio set 1	Portfolio set 2	Portfolio set 3	Portfolio set 1	Portfolio set 2	Portfolio set 3
100% Asset 1	0.0055	0.0023	0.0090	0.0038	0.0004	0.0086
90% Asset 1	0.0083*	0.0025	0.0692	0.0056*	0.0005	0.0658*
80% Asset 1	0.0077	0.0028	0.0673*	0.0050	0.0005	0.0637
70% Asset 1	0.0070	0.0031	0.0651	0.0044	0.0006	0.0610
60% Asset 1	0.0064	0.0035	0.0623	0.0038	0.0007	0.0578
50% Asset 1	0.0058	0.0040	0.0589	0.0032	0.0008	0.0538
40% Asset 1	0.0052	0.0046	0.0543	0.0027	0.0009	0.0485
30% Asset 1	0.0046	0.0056	0.0482	0.0021	0.0011	0.0414
20% Asset 1	0.0040	0.0069	0.0394	0.0016	0.0014	0.0312
10% Asset 1	0.0035	0.0091	0.0258	0.0010	0.0019	0.0155
0% Asset 1	0.0029	0.0132*	0.0016	0.0005	0.0028*	-0.0122

Notes: * states the best portfolio in the portfolio set for given risk-free rate.
Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).

Table 6: Second order Value at Risk portfolio's for the full sample period 04-01-2011 until 30-12-2020.

$\delta = 1/2487$	r = 1			r = 1.0015		
Weights	Portfolio set 1	Portfolio set 2	Portfolio set 3	Portfolio set 1	Portfolio set 2	Portfolio set 3
100% Asset 1	0.0055	0.0023	0.0090	0.0038	0.0004	0.0086*
90% Asset 1	0.0057	0.0026	0.0090	0.0039	0.0005	0.0086
80% Asset 1	0.0060	0.0029	0.0090	0.0039	0.0005	0.0085
70% Asset 1	0.0063	0.0033	0.0090	0.0040*	0.0006	0.0085
60% Asset 1	0.0067	0.0039	0.0090	0.0040	0.0008	0.0084
50% Asset 1	0.0069*	0.0047	0.0090	0.0039	0.0009	0.0083
40% Asset 1	0.0067	0.0059	0.0090	0.0035	0.0012	0.0081
30% Asset 1	0.0059	0.0079	0.0091	0.0027	0.0016	0.0078
20% Asset 1	0.0048	0.0112	0.0091	0.0019	0.0023	0.0073
10% Asset 1	0.0038	0.0142*	0.0094*	0.0011	0.0030*	0.0057
0% Asset 1	0.0029	0.0132	0.0016	0.0005	0.0028	-0.0122

Notes:* states the best portfolio in the portfolio set for given risk-free rate.
Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).

7.2 Extreme Risk Index

Like the safety first criterion, the ERI focuses on the downside risk of the assets. That is why we consider the losses of the six assets. We examine whether the ERI could outperform the VaR based on second order expansion. The ERI uses equal tail indices for computing the optimal $\gamma_{\omega,t}$ values per portfolio set. Because the ERI exploits the dependence between two assets, we now combine our portfolio sets based on related assets. So when comparing the portfolio's of the ERI and the VaR based on the second order expansion, each portfolio set consists of a combination of the two stocks, two bonds and two commodities. Table 16 in Appendix D provides the results of the estimated optimal weights for the different portfolio sets. For the in-sample period, we calculate the optimal ERI portfolio's and second order VaR portfolio's using the estimated optimal weights. The results are shown in Table 7.

Firstly, we compare the estimated in-sample portfolio values to the out-of-sample portfolio's based on the second order VaR so we consider VaR as the downside risk measure. For the portfolio set consisting of stocks, the table shows that the estimated ERI portfolio's perform better than the VaR portfolio's when using VaR as a downside risk measure. For the portfolio set consisting of bonds, the second order VaR portfolio's are estimated more accurate compared to the estimates of the ERI portfolio's. The ERI portfolio's also perform slightly better for the portfolio set consisting of commodities.

Secondly, we use the ERI as the downside risk measure. The estimated in-sample portfolio values are compared to the portfolio's based on the ERI. The results show that all ERI portfolio's outperform the VaR portfolio's when using the ERI as a downside risk measure. Based on the results, we can now conclude that in most cases the ERI portfolio's outperform the VaR portfolio's with second order expansion. Therefore, it's beneficial for an investor to use the ERI method for estimating the optimal portfolio using related asset, because it could lead to higher returns and more correct forecasting estimates.

Table 7: Optimal portfolio value per portfolio set based on Value at Risk and the Extreme Risk Index for an in-sample period from 04-01-2011 until 30-12-2017 and an out-of-sample period from 02-01-2018 until 30-12-2020.

	Portfolio set stocks	Portfolio set bonds	Portfolio set commodities
In-sample			
Value at Risk	0.0166	0.0025	0.0211
Extreme Risk Index	0.0248	0.0234	0.0262
Out-of-sample			
Value at Risk	0.0262	0.0025	0.0285
Extreme Risk Index	0.0480	0.0835	0.0930

Note: The Value at Risk is based on the second order expansion and computed using equation (4).

7.3 Backtesting

The backtesting procedure starts by calculating the second order VaR using equation (4) for a 99% level, q = 0.01. The results for the in-sample VaR values are shown in Table 18 in Appendix E. Using these VaR levels, we perform two violations tests to determine whether the second order expansion is more effective when estimating the VaR. Prior to testing, we define the indicator variables for each portfolio in every portfolio set. The indicator sums up the amount of times the VaR levels are violated for the returns in the out-of-sample period. Knowing the amount of violations we can

compute the log-likelihood functions, the values of the LR-test and the corresponding p-values of the χ^2-test for the unconditional coverage tests and the independence test.
in Table 8 shows the amount of violations per portfolio for the unconditional coverage test are shown by T_1 and the amount of non-violations per portfolio by T_0. It can be seen that for Portfolio set 1, we can reject the null-hypothesis of unconditional coverage for all portfolio's where the majority is invested in the S&P 500. However, a portfolio that contains only Gold or a mix of S&P 500 and Gold (60/40) accepts the null hypothesis of unconditional coverage. The LR value is not defined for three portfolio's in this portfolio set, because there were no violations of the VaR levels. For Portfolio set 2, three portfolio's, where the majority is invested in Corporate bonds, do not reject the null-hypothesis of unconditional coverage and one portfolio does not contain violations of the VaR level. Lastly, for Portfolio set 3 we can accept the null hypothesis of unconditional coverage for all portfolio's where the majority is invested in Government bonds.

Table 8: Non-violations and violations of the unconditional coverage test per portfolio for the out-of-sample period 02-01-2018 until 30-12-2020.

q = 0.01	Portfolio set 1			Portfolio set 2			Portfolio set 3		
Weights	T_0	T_1	$\hat{\pi}$	T_0	T_1	$\hat{\pi}$	T_0	T_1	$\hat{\pi}$
100% Asset 1	708	38	0.0509	729	17	0.0228	729	17	0.0228
90% Asset 1	708	38	0.0509	729	17	0.0228	729	17	0.0228
80% Asset 1	707	39	0.0523	729	17	0.0228	729	17	0.0228
70% Asset 1	707	39	0.0523	729	17	0.0228	729	17	0.0228
60% Asset 1	710	36	0.0483	730	16	0.0214	729	17	0.0228
50% Asset 1	721	25	0.0335	730	16	0.0214	729	17	0.0228
40% Asset 1	742	4	0.0054***	732	14	0.0188***	731	15	0.0201***
30% Asset 1	746	0	0.0000	732	14	0.0188***	731	15	0.0201***
20% Asset 1	746	0	0.0000	729	17	0.0228	732	14	0.0188***
10% Asset 1	746	0	0.0000	746	0	0.0000	735	11	0.0147***
0% Asset 1	739	7	0.0094***	736	10	0.0134***	743	3	0.0040***

Notes: Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).
The out-of-sample size is equal to 746.
*** represents the 1 % significance for the p-value of the χ^2-test corresponding to the LR-test

For the independence test, the results of the LR-test are disappointing. The amount of violations per Markov-transition and the corresponding probabilities are shown in Table 9 for Portfolio set 1. For each portfolio the null-hypotheses is rejected. From this we can conclude that the independence test is extremely dependent on the amount of violations of the VaR levels, and due to the fact that number of violations in the out-of-sample period is low and sometimes even zero, it leads to extremely low or non-existing probabilities and LR values, which can't perfectly explain the existence of independence for the assets in the portfolio's. Portfolio set 2 and 3 show the same results and can be seen in Table 19 and Table 20 in Appendix E. Christoffersen (1998) doesn't state an alternative approach or solution in case there are no violations. Therefore, it's not wise to accept the null hypothesis of independence for the given portfolio sets.

Table 9: Markov transitions and transition probabilities for the independence test per portfolio for portfolio set 1 for the out-of-sample period 02-01-2018 until 30-12-2020.

Weights	Portfolio set 1 T_{00}	T_{10}	T_{01}	T_{11}	$\hat{\pi_{01}}$	$\hat{\pi_{11}}$	$\hat{\pi_2}$
100% Asset 1	671	35	35	3	0.0496	0.0789	0.0511
90% Asset 1	673	34	34	3	0.0481	0.0811	0.0497
80% Asset 1	676	32	32	4	0.0452	0.1111	0.0484
70% Asset 1	676	32	32	4	0.0452	0.1111	0.0484
60% Asset 1	683	29	29	3	0.0407	0.0938	0.0430
50% Asset 1	697	22	22	3	0.0306	0.1200	0.0336
40% Asset 1	736	4	4	0	0.0054	0.0000	0.0054
30% Asset 1	744	0	0	0	0.0000	-	0.0000
20% Asset 1	744	0	0	0	0.0000	-	0.0000
10% Asset 1	744	0	0	0	0.0000	-	0.0000
0% Asset 1	731	6	6	1	0.0081	0.1429	0.0094

Notes: Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).
The out-of-sample size is equal to 746.
*** represents the 1 % significance for the p-value of the χ^2-test corresponding to the LR-test

Given that most portfolio's don't accept the null hypothesis of unconditional coverage and we can't define a correct independence test, because there are little to no violations of the VaR levels, we can state that the VaR estimates for the second order expansion are too conservative. Thus, we can conclude that the second order expansion is not more effective when estimating the VaR. There are a few portfolio's which do accept the null hypothesis of unconditional coverage. More importantly, having zero exceedances of the VaR levels is not defined for our used violation test. For these results we therefore can't give a proper answer on the effectiveness of the second order VaR.

8 Conclusion

In this paper, we compute safety first portfolio's corresponding to VaR levels based on the first order and second order expansion and ERI portfolio's for three portfolio sets of six different assets. Hyung & de Vries (2017) state that adding the second order expansion leads to more balanced portfolio's. Using backtesting and different violation tests, our goal is to investigate whether adding the second expansion is also more effective when estimating the VaR. We re-create the safety first portfolio's with first and second order expansion from Jansen et al (2000) and Hyung & de Vries using different assets, and examine whether the second order expansion indeed results in more a balanced solution for the used data. This paper also looks at another downside risk measures, the ERI, to investigate whether the ERI could outperform the safety first portfolio with second order expansion.

Firstly, we choose different portfolio sets with a combination of two assets based on their correlation. Using the estimated bootstrap vales and tail indices per asset, we compute the VaR levels based on

the first and second order expansion for different failure probabilities. The VaR levels are then used to estimate the safety first portfolio's. For the used portfolio sets and chosen failure probabilities, we can confirm the conclusion of Hyung & de Vries (2017). Portfolio's based on the second order expansion indeed lead on average to more balanced solutions, but portfolio's based on the first order do give slightly higher returns.
Secondly, we investigate whether portfolio's based on the ERI, another downside risk measure that uses equal tail indices and the dependence between two assets, could outperform VaR portfolio's with second order expansion. Using portfolio sets consisting of related assets, we can conclude that the ERI portfolio's outperform the VaR portfolio's based on the second order expansion when considering the ERI and VaR as downside risk measure. Lastly, we perform different violation tests to measure the effectiveness of the second order expansion. We estimate VaR levels based on Proposition 3 from Hying & de Vries (2017). From the unconditional coverage test can be concluded that most portfolio's of the portfolio sets don't accept the null hypothesis of unconditional coverage. Also, some portfolio's have no violations of the VaR levels so the unconditional coverage test and the independence test give poor results for these portfolio's. All other portfolio's reject the null hypothesis of independence. In conclusion, we can state that, because the two violation tests perform poorly, our estimated VaR values are too conservative and adding the second order expansion is not more effective when estimating the VaR. However, the violation tests are not defined when a portfolio holds zero exceedances. This could result in the VaR being too conservative, but we are unable to properly define the effectiveness of the second order VaR for theses cases.

This research deals with a few complications. During the volatile year of 2020, the returns of the asset Crude Oil became negative, giving unusual results for the logarithmic returns, the bootstrap values and the tail indices. Also, the violation tests couldn't be executed properly because some portfolio's hold zero exceendances of the VaR levels. For further research, other violation tests that do accept zero exceedances could be performed to estimate the effectiveness of the second order expansion. Also, the VaR measure doesn't meet the requirements of sub-additivity according to Acerbi & Tasche (2001). That is why one could also consider investigating safety first portfolio's using a more coherent risk measure such as the Expected Shortfall.

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Appendix A Statistical properties

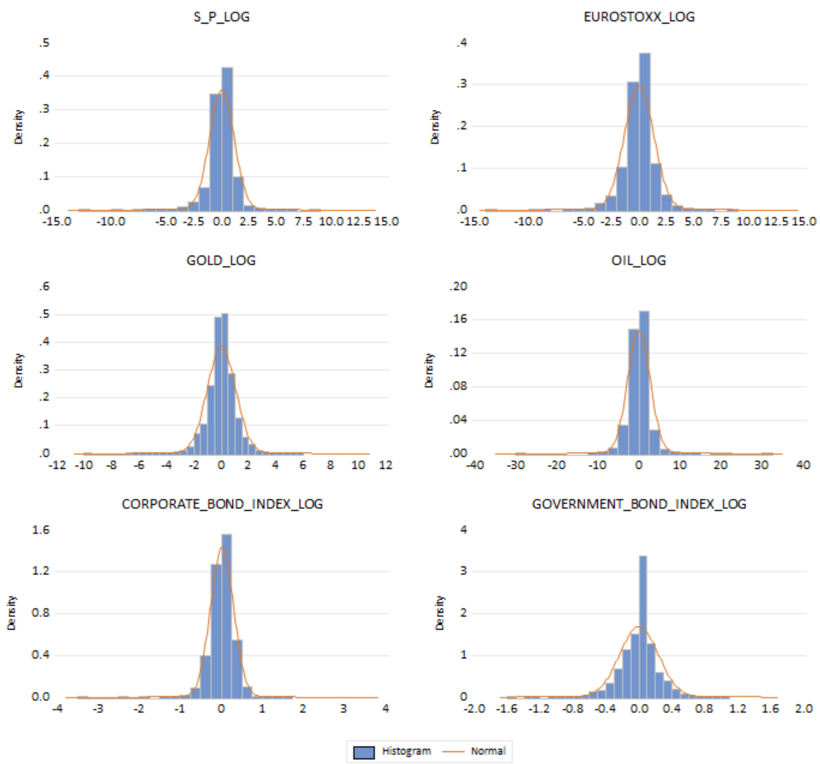

Figure 1: Histograms of the statistical properties of the logarithmic daily returns of the six assets for the data period 04-01-2011 until 30-12-2020.

Appendix B Bootstrap method

For the bootstrap method we sort the asset returns in ascending order, such that $X_{n,1} \leq ... \leq X_{n,n}$ are the order statistics of the asset returns. Following Danielsson et al. (2001) we again use the Hill estimator (1975):

$$\zeta_n(m) = \frac{1}{m}\sum_{i=1}^{m} log(X_{n-i+1}) - Log(X_{n-m}). \tag{30}$$

To determine the most appropriate order statistic m, we will estimate the asymptotic mean squared error (AMSE) of the Hill estimator ζ_n using the bootstrap procedure:

$$AMSE(n,m) = AsyE(\zeta_n(m) - \zeta)^2. \tag{31}$$

To find the optimal value of m the AMSE will be minimized. The value of ζ is unknown, but can be estimated by using the bootstrap re-sample method, where we choose a smaller sample size n_1, $n_1 \leq n$, drawn from sample size $X_n = X_1 .....X_n$. Danielsson et al. (2001) suggest that an appropriate value of n_1 is $n^{1-\epsilon}$, with $0 < \epsilon < \frac{1}{2}$. We again define the order statistics of the smaller sample size by $X_{n1,1} \leq ... \leq X_{n1,n1}$ and define the Hill estimate for the smaller sample size:

$$\zeta_{n_1}(m_1) = \frac{1}{m_1}\sum_{i=1}^{m_1} log(X_{n_1-i+1}) - Log(X_{n_1-m_1}). \tag{32}$$

By using the bootstrap estimate we can compute the mean squared error of the sample:

$$Q(n_1, m_1) = E((M_{n1}(m_1) - 2(\zeta_{n1}(m1))^2)^2), \tag{33}$$

$$\text{where} \quad M_{n1}(m_1) = \frac{1}{m_1}\sum_{i=1}^{m_1} log(\frac{(X_{n1} - i + 1)}{(X_{n1} - m)^2}). \tag{34}$$

From the equation we take the value of m_1 that minimizes the function $Q(n_1, m_1)$. We will repeat the same procedure for a smaller sample size, where we choose the same smaller sample size, $n_2 = \frac{n_1^2}{n}$, as Danielsson et al. (2016). We obtain the value m_2 which minimizes $Q(n_2, m_2)$. Then the optimal value of m for the order statistic can be computed in the following manner:

$$\hat{m} = \frac{m_2^2}{m_1}[\frac{log(m_1)^2}{(2log(n_1) - log(m_1))^2}]^{\frac{log(n_1)-log(m_1)}{log(n_1)}}. \tag{35}$$

Appendix C Safety first portfolio's

Value at Risk levels

Table 10: Value at Risk levels of to the first order and second order expansion corresponding to the determined probability for the full sample period 04-01-2011 until 30-12-2020.

$\delta = 0.75/2487$	**First order**			**Second order**		
Weights	Portfolio set 1	Portfolio set 2	Portfolio set 3	Portfolio set 1	Portfolio set 2	Portfolio set 3
100% Asset 1	-0.1005	-0.0843	-0.4665	-0.1005	-0.0843	-0.4665
90% Asset 1	-0.0603	-0.0773	-0.0492	-0.0905	-0.0759	-0.4198
80% Asset 1	-0.0611	-0.0705	-0.0449	-0.0805	-0.0674	-0.3732
70% Asset 1	-0.0618	-0.0636	-0.0407	-0.0706	-0.0590	-0.3265
60% Asset 1	-0.0625	-0.0567	-0.0365	-0.0613	-0.0506	-0.2799
50% Asset 1	-0.0633	-0.0498	-0.0322	-0.0538	-0.0422	-0.2332
40% Asset 1	-0.0640	-0.0429	-0.0280	-0.0499	-0.0338	-0.1866
30% Asset 1	-0.0647	-0.0360	-0.0238	-0.0506	-0.0255	-0.1399
20% Asset 1	-0.0655	-0.0291	-0.0195	-0.0546	-0.0179	-0.0933
10% Asset 1	-0.0662	-0.0223	-0.0153	-0.0604	-0.0143	-0.0466
0% Asset 1	-0.0669	-0.0154	-0.0111	-0.0669	-0.0154	-0.0111

Notes: Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).

Table 11: Value at Risk levels of to the first order and second order expansion corresponding to the determined probability for the full sample period 04-01-2011 until 30-12-2020.

$\delta = 0.5/2487$	**First order**			**Second order**		
Weights	Portfolio set 1	Portfolio set 2	Portfolio set 3	Portfolio set 1	Portfolio set 2	Portfolio set 3
100% Asset 1	-0.1169	-0.0931	-0.5747	-0.1169	-0.0931	-0.5747
90% Asset 1	-0.0672	-0.0854	-0.0524	-0.1052	-0.0838	-0.5172
80% Asset 1	-0.0680	-0.0778	-0.0479	-0.0936	-0.0745	-0.4597
70% Asset 1	-0.0688	-0.0702	-0.0434	-0.0821	-0.0652	-0.4023
60% Asset 1	-0.0696	-0.0626	-0.0388	-0.0711	-0.0559	-0.3448
50% Asset 1	-0.0705	-0.0550	-0.0343	-0.0620	-0.0466	-0.2873
40% Asset 1	-0.0713	-0.0474	-0.0298	-0.0568	-0.0373	-0.2299
30% Asset 1	-0.0721	-0.0398	-0.0253	-0.0568	-0.0281	-0.1724
20% Asset 1	-0.0729	-0.0322	-0.0208	-0.0609	-0.0198	-0.1149
10% Asset 1	-0.0737	-0.0246	-0.0163	-0.0672	-0.0157	-0.0575
0% Asset 1	-0.0745	-0.0170	-0.0118	-0.0745	-0.0170	-0.0118

Notes: Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).

Value at Risk portfolio's

First order

Table 12: First order Value at Risk portfolio's for the full sample period 04-01-2011 until 30-12-2020.

$\delta = 0.75/2487$	r = 1			r = 1.0015		
Weights	Portfolio set 1	Portfolio set 2	Portfolio set 3	Portfolio set 1	Portfolio set 2	Portfolio set 3
100% Asset 1	0.0049	0.0021	0.0077	0.0035	0.0004	0.0074
90% Asset 1	0.0077*	0.0023	0.0661*	0.0052*	0.0004	0.0629*
80% Asset 1	0.0071	0.0026	0.0644	0.0047	0.0005	0.0609
70% Asset 1	0.0065	0.0029	0.0622	0.0041	0.0006	0.0584
60% Asset 1	0.0059	0.0032	0.0596	0.0035	0.0006	0.0553
50% Asset 1	0.0053	0.0037	0.0563	0.0030	0.0007	0.0514
40% Asset 1	0.0048	0.0043	0.0519	0.0025	0.0009	0.0464
30% Asset 1	0.0043	0.0052	0.0461	0.0020	0.0011	0.0396
20% Asset 1	0.0037	0.0064	0.0377	0.0015	0.0013	0.0299
10% Asset 1	0.0032	0.0085*	0.0246	0.0010	0.0018	0.0148
0% Asset 1	0.0027	0.0123	0.0016	0.0005	0.0027*	-0.0117

Notes: * states the best portfolio in the portfolio set for given risk-free rate.
Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).

Table 13: First order Value at Risk portfolio's for the full sample period 04-01-2011 until 30-12-2020.

δ = 0.5/2487	r = 1			r = 1.0015		
Weights	Portfolio set 1	Portfolio set 2	Portfolio set 3	Portfolio set 1	Portfolio set 2	Portfolio set 3
100% Asset 1	0.0042	0.0019	0.0063	0.0030	0.0003	0.0060
90% Asset 1	0.0069*	0.0021	0.0621*	0.0047*	0.0004	0.0591*
80% Asset 1	0.0064	0.0023	0.0604	0.0042	0.0004	0.0571
70% Asset 1	0.0058	0.0026	0.0584	0.0037	0.0005	0.0548
60% Asset 1	0.0053	0.0029	0.0559	0.0032	0.0006	0.0519
50% Asset 1	0.0048	0.0034	0.0528	0.0027	0.0007	0.0483
40% Asset 1	0.0043	0.0039	0.0488	0.0022	0.0008	0.0436
30% Asset 1	0.0038	0.0047	0.0433	0.0018	0.0010	0.0372
20% Asset 1	0.0033	0.0058	0.0354	0.0013	0.0012	0.0280
10% Asset 1	0.0029	0.0077	0.0231	0.0009	0.0016	0.0139
0% Asset 1	0.0024	0.0111*	0.0015	0.0004	0.0024*	-0.0110

Notes: * states the best portfolio in the portfolio set for given risk-free rate.
Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).

Second order

Table 14: Second order Value at Risk portfolio's for the full sample period 04-01-2011 until 30-12-2020.

δ = 0.75/2487	r = 1			r = 1.0015		
Weights	Portfolio set 1	Portfolio set 2	Portfolio set 3	Portfolio set 1	Portfolio set 2	Portfolio set 3
100% Asset 1	0.0049	0.0021	0.0077	0.0035	0.0004	0.0074*
90% Asset 1	0.0051	0.0024	0.0077	0.0035	0.0004	0.0074
80% Asset 1	0.0054	0.0027	0.0077	0.0035	0.0005	0.0073
70% Asset 1	0.0057	0.0031	0.0078	0.0036	0.0006	0.0073
60% Asset 1	0.0060	0.0036	0.0078	0.0036*	0.0007	0.0072
50% Asset 1	0.0063*	0.0044	0.0078	0.0035	0.0009	0.0071
40% Asset 1	0.0061	0.0055	0.0078	0.0032	0.0011	0.0070
30% Asset 1	0.0054	0.0073	0.0078	0.0025	0.0015	0.0068
20% Asset 1	0.0045	0.0105	0.0079	0.0017	0.0022	0.0063
10% Asset 1	0.0035	0.0132*	0.0081*	0.0011	0.0028*	0.0049
0% Asset 1	0.0027	0.0123	0.0016	0.0005	0.0027	-0.0117

Notes: * states the best portfolio in the portfolio set for given risk-free rate.
Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).

Table 15: Second order Value at Risk portfolio's for the full sample period 04-01-2011 until 30-12-2020

δ = 0.5/2487	r = 1			r = 1.0015		
Weights	Portfolio set 1	Portfolio set 2	Portfolio set 3	Portfolio set 1	Portfolio set 2	Portfolio set 3
100% Asset 1	0.0042	0.0019	0.0063	0.0030	0.0003	0.0060*
90% Asset 1	0.0044	0.0022	0.0063	0.0030	0.0004	0.0060
80% Asset 1	0.0046	0.0024	0.0063	0.0030	0.0005	0.0060
70% Asset 1	0.0049	0.0028	0.0063	0.0031	0.0005	0.0059
60% Asset 1	0.0052	0.0033	0.0063	0.0031	0.0006	0.0059
50% Asset 1	0.0055*	0.0040	0.0063	0.0031*	0.0008	0.0058
40% Asset 1	0.0054	0.0050	0.0063	0.0028	0.0010	0.0057
30% Asset 1	0.0048	0.0066	0.0064	0.0022	0.0014	0.0055
20% Asset 1	0.0040	0.0095	0.0064	0.0016	0.0020	0.0051
10% Asset 1	0.0032	0.0120*	0.0066*	0.0010	0.0025*	0.0040
0% Asset 1	0.0024	0.0111	0.0015	0.0004	0.0024	-0.0110

Notes: * states the best portfolio in the portfolio set for given risk-free rate.
Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).

Appendix D Extreme Risk Index

Table 16: Optimal weights per asset of the portfolio set based on Value at Risk and the Extreme Risk Index for an in-sample period from 04-01-11 until 30-12-2017 and an out-of-sample period from 02-01-2018 until 30-12-2020.

	Portfolio set stocks		Portfolio set bonds		Portfolio set commodities	
	SP	EURO	Corporate	Government	Gold	Crude Oil
In-sample						
Value at Risk	0.6471	0.3529	0.2836	0.7164	0.7959	0.2041
Extreme Risk Index	0.5749	0.4250	0.4932	0.5068	0.5802	0.4198
Out-of-sample						
Value at Risk	0.5749	0.4250	0.4932	0.5068	0.5802	0.4198
Extreme Risk Index	0.5687	0.4313	0.1115	0.8885	0.9956	0.0044

Note:* Corporate stands for Corporate bond and Government stands for Government bond.

Appendix E Backtesting

Table 17: Bootstrap values and tail indices corresponding to the six assets during the in-sample period 04-01-2011 until 30-12-2017.

	α	m	n	X_m
S&P	3.6344	16	1741	0.0237
Gold	3.9847	17	1741	0.0259
EURO STOXX	3.7501	15	1741	0.0341
Corporate*	122.2574	1	1741	0.0076
Crude Oil	6.8086	7	1741	0.0844
Government*	4.7173	52	1741	0.0045

Notes: α represents the first order tail index, n is the number of observations and m is optimal order statistic value.
* Corporate stands for Corporate bond and Government stands for Government bond.

Table 18: Value at Risk levels for the in-sample period 04-01-2011 until 30-12-2017.

(q = 0.01)			
Weights	Portfolio set 1	Portfolio set 2	Portfolio set 3
100% Asset 1	-0.0232	-0.0328	-0.0738
90% Asset 1	-0.0209	-0.0295	-0.0664
80% Asset 1	-0.0186	-0.0262	-0.0590
70% Asset 1	-0.0165	-0.0230	-0.0517
60% Asset 1	-0.0151	-0.0197	-0.0443
50% Asset 1	-0.0164	-0.0164	-0.0369
40% Asset 1	-0.0287	-0.0131	-0.0295
30% Asset 1	-0.0919	-0.0098	-0.0221
20% Asset 1	-0.4894	-0.0066	-0.0148
10% Asset 1	-6.1383	-0.0000	-0.0076
0% Asset 1	-0.0257	-0.0074	-0.0057

Notes: Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).

Table 19: Markov transitions and transition probabilities for the independence test per portfolio for portfolio set 2 for the out-of-sample period 02-01-2018 until 30-12-2020.

	Portfolio set 2						
Weights	T_{00}	T_{10}	T_{01}	T_{11}	$\hat{\pi_{01}}$	$\hat{\pi_{11}}$	$\hat{\pi_2}$
100% Asset 1	712	15	15	2	0.0206	0.1176	0.0228
90% Asset 1	712	15	15	2	0.0206	0.1176	0.0228
80% Asset 1	712	15	15	2	0.0206	0.1176	0.0228
70% Asset 1	712	15	15	2	0.0206	0.1176	0.0228
60% Asset 1	713	15	15	1	0.0206	0.0625	0.0215
50% Asset 1	715	14	14	1	0.0192	0.0667	0.0202
40% Asset 1	719	12	12	1	0.0164	0.0769	0.0175
30% Asset 1	721	11	11	1	0.0150	0.0833	0.0161
20% Asset 1	724	9	9	2	0.0123	0.1818	0.0148
10% Asset 1	744	0	0	0	0.0000	-	0.0000
0% Asset 1	734	5	5	0	0.0068	0.0000	0.0067

Notes: Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).
The out-of-sample size is equal to 746.
*** represents the 1 % significance for the p-value of the χ^2-test corresponding to the LR-test.

Table 20: Markov transitions and transition probabilities for the independence test per portfolio for portfolio set 3 for the out-of-sample period 02-01-2018 until 30-12-2020..

Weights	Portfolio set 3 T_{00}	T_{10}	T_{01}	T_{11}	$\hat{\pi_{01}}$	$\hat{\pi_{11}}$	$\hat{\pi_2}$
100% Asset 1	713	14	14	3	0.0065	0.1765	0.0078
90% Asset 1	713	14	14	3	0.0065	0.1765	0.0078
80% Asset 1	713	14	14	3	0.0065	0.1765	0.0078
70% Asset 1	713	14	14	3	0.0065	0.1765	0.0078
60% Asset 1	713	14	14	3	0.0065	0.1765	0.0078
50% Asset 1	713	12	12	4	0.0056	0.2500	0.0074
40% Asset 1	716	10	10	4	0.0046	0.2857	0.0064
30% Asset 1	720	10	10	4	0.0046	0.2857	0.0064
20% Asset 1	721	10	10	3	0.0046	0.2308	0.0059
10% Asset 1	726	8	8	2	0.0037	0.2000	0.0046
0% Asset 1	740	2	2	0	0.0009	0.0000	0.0009

Notes: Asset 1 represents for Portfolio set 1 the S&P 500, for Portfolio set 2 EURO STOXX and for Portfolio set 3 Crude oil.
The remainder of the weight contains Asset 2. (e.g., in the second row the portfolio contains 10% of Asset 2).
The out-of-sample size is equal to 746.
*** represents the 1 % significance for the p-value of the χ^2-test corresponding to the LR-test

Appendix F Programming code

For this research, we used several coding programs which can be found in the attached ZIP-file. The data set, used for the programs, exist of six assets, which are inserted in the programs in the following order from left to right: S&P 500, Gold, EURO STOXX, Corporate bond, Crude Oil and Government bond. We used e-views to compute the statistical properties and the correlations of the daily returns (no code required). We used R with packages tea and tidyverse to estimate the bootstrap values m. The R code for the bootstrap values can be found in the ZIP-file under the name: "mBootstrapmethod". The estimated values of m from the bootstrap method are then used in the matlab codes. The file "VaR_first_order" in the ZIP-file is a matlab code that computes the first order VaR quantiles and safety first portfolio's for the three established portfolio sets. The file "VaR_second_order" in the -file is a matlab code that computes the second order VaR quantiles and safety first portfolio's for the three established portfolio sets. The file "ERI_optimization" in the ZIP-file is a matlab code that computes the optimal weights and optimal portfolio's based on second order VaR using proposition 3 from Hyung & de Vries (2007) and the optimal weights and portfolios based on the ERI. The file "backtesting" in the ZIP-file is a matlab code that computes the VaR using proposition 3 from Hyung & de Vries (2007) for the in-sample period, the unconditional coverage test and the independence test.