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An Extended Mixed-Integer Programming Formulation for the Stochastic Lot- Sizing Problem extended with a Hybrid Model combining the Back-order and Lost Sales variants

Lotte Struik (524529)

Supervisor:	Heuvel, W van den
Second assessor:	Wagelmans, APM
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Contents

Abstract

We present a study that replicates and extends the research conducted by Tunc, Kilic, Tarim and Rossi (2018) in their paper titled "An Extended MIP Formulation and Dynamic Cut Generation Approach for the Stochastic Lot-Sizing Problem". The stochastic lot-sizing problem, which arises due to uncertainty in demand, is a significant issue in the field of operations management. Tunc et al. (2018) developed an extended mixed-integer programming (MIP) formulation which accommodates various adaptations of the problem. They employed the static-dynamic uncertainty strategy by implementing a predefined piecewise linear approximation of the cost function, and demonstrated that their formulation is more time-efficient compared to other existing models in literature. In this study, we first validate the findings of Tunc et al. (2018) by replicating their methodology. Subsequently, we propose an extension to the original model by introducing a hybrid version that accommodates both back-ordered and lost sales scenarios for unsatisfied demand, thereby enhancing the realism and applicability of the model. This extension aims to provide decision-makers with a more versatile tool for addressing the challenges posed by demand uncertainty.

1 Introduction

The lot-sizing problem is central to inventory management, with the primary objective being to develop a cost-effective inventory plan that minimizes the total cost while meeting demand over a specified finite time horizon. Previously, lot-sizing problems were studied under the assumption of deterministic demand, meaning that the demand was assumed to be known in advance, with certainty. However, in practical settings, demand is often unpredictable as it is inherently stochastic, and can be more accurately represented as a random variable. This shift from the deterministic to stochastic characterization of demand introduces a new layer of complexity to the lot-sizing problem. Specifically, due to demand uncertainty, inventory levels and future costs are also considered as random variables. As such, contemporary research primarily concentrates on the stochastic interpretation of the lot-sizing problem. It is important to clarify that with stochastic demand, the result of a lot-sizing problem doesn't yield a predetermined inventory plan, but instead, an inventory policy. This policy defines the rules for decision-making, indicating whether an order should be placed based on the current period and inventory levels, and if so, outlining the quantity to order (Tunc et al., 2018).

One approach to constructing such an inventory policy is through the use of the static-dynamic uncertainty strategy, as introduced by Bookbinder and Tan (1988). In this strategy, two components are involved in decision-making: a static component and a dynamic component. The *static* aspect refers to the scheduling of order timings, which are predetermined at the beginning of the planning horizon. This means that the moments at which orders will be placed are fixed in advance. Conversely, the *dynamic* aspect relates to the quantity of each order, which is determined adaptively at the beginning of each replenishment cycle based on current inventory levels and anticipated future demand. This dynamic decision-making allows the strategy to respond to real-time conditions. Özen, Doğru and Tarim (2012) note that within the static-dynamic uncertainty strategy, the optimal order quantity typically adheres to a base-stock policy. The essential idea of a base-stock policy is to maintain a constant number of items (referred to as the base-stock level) in the inventory. When items are removed from inventory to meet demand, an order is placed to restore the stock to this predetermined base-stock level, with the quantity ordered depending on the replenishment cycle's schedule and current conditions.

As mentioned in Tunc et al. (2018), this approach appears to be appealing because it allows for better coordination among supply chain players and facilitates joint replenishment and shipment consolidations. It strikes a balance between having a pre-determined schedule (providing predictability and allowing coordination) and being able to adapt to changing conditions by adjusting order quantities as necessary.

Addressing the stochastic lot-sizing problem is crucial and relevant in this day and age for a variety of reasons. Firstly, it enhances the decision-making process by integrating the uncertainty and unpredictability into business operations. This allows for businesses to have a more realistic perspective and make more effective choices when it comes to their demand and necessary inventory levels. Additionally, it helps businesses with risk management as it allows them to balance the costs and benefits associated with overstocking and under-stocking. Ultimately, companies that manage to properly manage their inventory, taking into account uncertain demand, can gain a significant competitive edge, fulfil customer demand more effectively and improve their profitability.

There are two primary approaches to solving the stochastic lot-sizing problem. The first approach involves developing customized algorithms that are designed explicitly for specific variations of the problem. The second approach, which is explored by Tunc et al. (2018) and is the focus of this thesis, employs mathematical programming models that can be solved using standard solvers. This latter approach is typically more generalized, making it applicable to a broader range of problem variations. However, a challenge associated with using this approach is addressing the nonlinear nature of the cost function. (Tunc et al., 2018) tackled this challenge by evaluating two methods. The first method substitutes the nonlinear cost function with an a predefined piecewise linear function, aligning with traditional methodologies found in previous academic literature. The second method utilizes a dynamic cut generation approach, which does not require a prior approximation and dynamically constructs the cost function during the solution process. In this thesis, the replication and extension will concentrate on the variant of the model that employs a piecewise linear approximation to address the nonlinear cost function.

We propose an extension to the extended MIP formulation by developing a hybrid version of the model. This hybrid model allows for a portion of the unsatisfied demand, or shortages, to be back-ordered, while the remainder is considered as lost sales. This aims to more closely mirror real-world scenarios, where some customers are willing to wait for back-ordered items, while others may choose alternative options, such as purchasing a substitute product or turning to a competitor. Although mixed models incorporating both back-orders and lost sales have been explored in prior research, such as by Castellano (2015), there is a scarcity of studies that have employed the static-dynamic uncertainty strategy in this context. Our contribution primarily lies in utilizing this approach and integrating the hybrid model with the extended formulation developed by Tunc et al. (2018).

The structure of this thesis is as follows: we will begin with a literature review to contextualize both this thesis and the work of Tunc et al. (2018). This will be followed by a definition of the problem statement for both of the separate original models using back-orders and lost sales. Subsequently, the methodology, and the results obtained from replicating the findings of Tunc et al. (2018) and the suggested extension, will be presented. The results obtained from the formulations will be compared and checked by means of simulation. Lastly, all findings and comparisons will be evaluated and concluded.

2 Literature Review

The stochastic lot-sizing problem (SLSP) has been a critical area of research within production planning and supply chain management for several years. Tunc et al. (2018) aim to offer a more efficient method in comparison to previous approaches to tackle the computational challenges associated with the problem while maintaining the solution's quality.

Prior to Tunc et al. (2018)'s work, several researchers have attempted to address the SLSP. One of the early papers on the subject is by Wagner and Whitin (1958), who introduced the deterministic lot-sizing problem (DLSP). They proposed a dynamic programming algorithm to solve the DLSP, which provided a foundation for future research in lot-sizing. However, the model by Wagner and Whitin (1958) was a deterministic model and assumed perfect knowledge of future demands, which is often unrealistic in real-world applications. Bookbinder and Tan (1988) were the first to provide a variation of strategies that could be used to solve the SLSP: static-dynamic uncertainty, static uncertainty and dynamic uncertainty. As mentioned, under the conditions of the static-dynamic uncertainty strategy, the timing of all orders has been fixed in advance. However, the actual lot sizes are determined only after the demand in preceding periods is known.

After the publication of the static-dynamic uncertainty strategy, many have used this approach and presented solutions using this strategy. Tarim and Kingsman (2004) introduced a mixed integer linear programming model (MILP) which simultaneously determined both the ordering periods and the base-stock levels while adhereing to α-service level constraints. This constraint measures the likelihood that the demand for a particular time span will be satisfied by the quantity of products available at the beginning of that same time span. Tunc et al. (2018) also mention a variant of their model the integrates service level constraints, however, this will not be further explored in this thesis. Tempelmeier (2007) further extends this model by incorporating the possibility of negative inventory levels. Additionally, Rossi, Kilic and Tarim (2015) propose MILP models that establish base-stock levels for the single-product stochastic uncapacitated lot-sizing problem under various service levels, utilizing a shortage-cost model. In their prior work, they introduce a linearization technique that provides upper and lower bounds for the first-order loss function (Rossi, Tarim, Prestwich & Hnich, 2014). The lower bounds that they presented were also used by Tunc et al. (2018) and will be further explored in this thesis. The static-dynamic uncertainty strategy is also the one that Tunc et al. (2018) have applied in their paper. Despite the advances made using the static-dynamic uncertainty strategy, computational complexity remained a significant challenge. The extended MIP formulation and dynamic cut generation approach in Tunc et al. (2018) builds on this earlier work, aiming to overcome the limitations of existing methods. For example, they state their model to be 'computationally far superior to the state-of-the-art MIP formulations in the literature', like those by Tarim and Kingsman (2006) and Rossi et al. (2015).

Many subsequent papers have used the work and model created by Tunc et al. (2018). For example, the paper by Gruson, Cordeau and Jans (2021) addresses stochastic three-level lot sizing and replenishment problem and uses Tunc et al. (2018)'s demand distributions to build their solution. Additionally, the paper by the same author, Tunc (2021) looks at the capacitated stochastic lot-sizing problem integrated with controllable processing times with convex compression cost and builds on the dynamic cut generation approach of Tunc et al. (2018) to similarly bypass the piecewise linear approximation in their MIP formulation.

The main additional contribution of this thesis will be the extension of creating a hybrid version of the model. A mixture of both back-ordering and lost sales as a way of considering shortages can be found in prior literature. Among the earlier literature, Montgomery, Bazaraa and Keswani (1973) created an inventory model that accounted for constant demand and partial backordering over an infinite time period. The proposed a method for solving the model by first transforming the cost function to be non-singular and then minimizing it in two stages. They applied this in scenarios of deterministic and heuristic-based stochastic demand.

A more recent example is the paper by Castellano (2015), they used a reorder point-lot size (r,Q) inventory model under stochastic demand and included a back-orders-lost sales mixture to deal with shortages. In their paper shortages are partially back-ordered with a fraction (β) while a fraction $(1 - \beta)$ of shortages is lost. They reasonably assume that β is supposed to be known by the decision maker. A similar approach to will be used in our methodology.

3 Problem definition & Preliminaries

In addressing the stochastic lot-sizing problem, the primary objective is to devise an optimal replenishment schedule that minimizes the expected total cost over a specified planning horizon. This schedule should outline the suitable periods for replenishment and determine the ideal base-stock levels for each period, adhering to the static-dynamic uncertainty strategy. Two variations of this problem are central to this thesis: firstly, a scenario assuming that all unsatisfied demand is back-ordered, and secondly, a scenario where any unsatisfied demand is considered a lost sale. These two scenarios are crucial to the extension proposed in the thesis and will also be the focal point of this replication section.

To address the complexity of the problem, an extended Mixed Integer Programming (MIP) model is employed. This model, based on the formulation proposed by Tunc et al. (2018), is rooted in the principle of calculating the costs linked to the replenishment cycle within designated time frames. The MIP model is crafted to simultaneously determine the optimal replenishment periods and establish the corresponding optimal base-stock levels. In doing so, the model aims to minimize the expected total cost over the planning horizon. Below the models for the back-ordering, lost sales and proposed hybrid version can be found.

The following details and parameters are applicable to all three models being considered. As mentioned in Tunc et al. (2018), the problem is set within a finite planning horizon composed of N discrete time periods. Let D_{ij} represent the random demand during the time interval $[i, j]$. The replenishment schedule is established at the outset of the planning horizon. Suppose there are m replenishment periods, denoted as $T_1, ..., T_m$, where m represents the total number of scheduled replenishment instances. It's assumed that the first period, T_1, marks the beginning of the first cycle, and an additional time point $T_{m+1} = N + 1$ marks the end of the final cycle. Consequently, the planning horizon can be viewed as a disjoint union comprising m replenishment cycles.

Regarding the decision variables, $x_{ij} \in \{0, 1\}$ are indicator variables that take the value 1 if $[i, j)$ is a replenishment cycle, and 0 otherwise, where $i, j \in [1, N + 1]$. Additionally, $H_{ijt} \in \mathbb{R}^n_+$

denote the approximate loss function values at period t of the replenishment cycle $[i, j)$. This function is bounded from below by constraints so that it abides the piecewise linear approximation of the loss function. As described in Tunc et al. (2018), the loss function evaluated at the base-stock level is $L_{it}(q_{ij} - \mathbb{E}D_{1i-1}) \approx \max_{(a,b)\in W_{it}}\{a + b(q_{ij} - \mathbb{E}D_{1i-1})\}$. For this, $W_{it} \subseteq \mathbb{R}^2$ denotes the set of intercept and slope pairs, (a, b), defining the piecewise linear approximation of the loss function. How this set is created is further explained in the methodology section.

Concerning the cost components, there are three elements to consider: a fixed cost, denoted by K, incurred for each order; a holding cost, denoted by h, charged per unit of inventory carried over from one period to the next; and a back-order cost, denoted by p, charged per unit of back-ordered demand. For the lost sales model we are dealing with a lost sale cost, denoted by v. With all the general preliminaries explained, the next subsections will define the full MIPs.

3.1 Back-order variant

The first version of the problem that is assessed is the one where it is assumed that the demands that arrive when there is no stock left are back-ordered and as such, incur a back-order cost. To be able to build the model for this situation, the decision variable q_{ij} is introduced. The variable $q_{ij} \in \mathbb{R}^n_+$ represents the expected cumulative order quantities up to and including period i, given that $[i, j)$ is a replenishment cycle, and 0 otherwise.

The MIP problem can be formally defined as follows:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{N}\sum_{j=i+1}^{N+1}(Kx_{ij} + \sum_{t=i}^{j-1}(h(q_{ij} - \mathbb{E}D_{1t}x_{ij}) + (h+p)H_{ijt})) \\
\text{subject to} \quad & \sum_{i=1}^{t-1} x_{it} = \sum_{j=t+1}^{N+1} x_{tj}, && t \in [2, N], && (1) \\
& \sum_{j=2}^{N+1} x_{1j} = 1, && && (2) \\
& \sum_{i=1}^{N} x_{i,N+1} = 1, && && (3) \\
& q_{ij} \le Mx_{ij}, \\
& \quad i \in [1, N], j \in [i+1, N+1] && && (4) \\
& \sum_{i=1}^{t-1} q_{it} \le \sum_{j=t+1}^{N+1} q_{tj}, && t \in [2, N], && (5) \\
& H_{ijt} \ge ax_{ij} + b(q_{ij} - \mathbb{E}D_{1i-1}x_{ij}), \\
& \qquad i \in [1, N], j \in [i+1, N+1], (a, b) \in W_{it} && && (6)
\end{aligned}
$$

The first term in the objective function represents the fixed ordering cost, which is incurred whenever an order is placed. The second term represents the holding and shortage costs, which are incurred due to inventory holding and shortages which are back-ordered, respectively.

The initial three constraints ensures the proper functioning of the problem and that all periods and cycles occur as they are intended. Specifically, constraint (1) makes sure that when a replenishment cycle starts at a given time period then another must end at the same time period. Constraint (2), on the other hand, guarantees that the inaugural replenishment cycle kicks off at the start of the first time period. Similarly, constraint (3) is in place to ensure that the final replenishment cycle wraps up at the conclusion of the last time period.

Constraint (4), where M is a sufficiently large number, makes sure that the expected cumulative order quantity (q_{ij}) can only be positive if $[i, j)$ is an established replenishment cycle, hence if x_{ij} is equal to 1. Additionally to this, the cumulative order quantities should be non-decreasing from one replenishment cycle to the next. Hence the difference between cumulative order quantities of a cycle and its successive cycle should always be non-negative. This is ensured by the fifth constraint (5). Lastly, constraint (6) makes sure that the approximate loss function value at period t of the replenishment cycle (H_{ijt}) is larger than the value of the loss function evaluated at the base-stock level, dependent on whether or not $[i, j)$ is an established replenishment cycle.

3.2 Lost Sales variant

The second variation of the problem that we examine involves treating demands that materialize when the system is out of stock as lost sales instead of back-orders. This implies that rather than incurring a back-ordering cost, the system suffers a loss in revenue due to the missed sale.

The fundamental structure of the MIP model remains intact, with the key modification being the substitution of the back-ordering cost p with a penalty cost v for each lost sale, where v represents the unit cost of an item. Furthermore, adjustments are made to the inventory balance dynamics to account for the fact that the expected inventory level is now equivalent to the expected on-hand stock. To facilitate this, a new decision variable $s_{ij} \in \mathbb{R}^n_+$ is introduced, representing the base-stock level in period i if $[i, j)$ is a replenishment cycle. With these modifications, the MIP model can be reformulated as follows:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{N}\sum_{j=i+1}^{N+1}(Kx_{ij} + \sum_{t=i}^{j-1} h(s_{ij} - \mathbb{E}D_{it}x_{ij} + H_{ijt}) + vH_{ijj-1}) && \\
\text{subject to} \quad & \sum_{i=1}^{t-1} x_{it} = \sum_{j=t+1}^{N+1} x_{tj}, & t \in [2, N], \quad & (7) \\
& \sum_{j=2}^{N+1} x_{1j} = 1, & & (8) \\
& \sum_{i=1}^{N} x_{i,N+1} = 1, & & (9) \\
& s_{ij} \le Mx_{ij}, & & \\
& \quad i \in [1, N], j \in [i+1, N+1] & & (10) \\
& \sum_{i=1}^{t-1}(s_{it} - \mathbb{E}D_{it-1}x_{it} + H_{itt-1}) \le \sum_{j=t+1}^{N+1} s_{tj}, & t \in [2, N], \quad & (11) \\
& H_{ijt} \ge ax_{ij} + bs_{ij}, & & \\
& \quad i \in [1, N], j \in [i+1, N+1], (a, b) \in W_{it} & & (12)
\end{aligned}
$$

In this model the first three constraints are equivalent to first three in the back-ordering variant of the MIP. The first new constraint for this model (7), where M is a sufficiently large number, ensures that the base-stock level s_{ij} should be zero if $[i, j)$ is not a replenishment cycle. The next constraint (8) makes sure that at the end of a replenishment cycle, the expected stock on-hand cannot be more than the base-stock level of the next replenishment cycle. Finally, again the last constraint (9) makes sure that the approximate loss function value at period t of the replenishment cycle (H_{ijt}) is larger than the value of the loss function evaluated at the base-stock level

3.3 Hybrid variant

To develop a hybrid MIP model that integrates both variants, the lost sales and the back-ordering for shortages, it is essential to first asses the differences between them and how the models work.

The back-ordering variant employs the variable q_{ij}, representing the cumulative order quantities up to and inclusive of period i. The costs in this model are computed by evaluating the holding quantity (H_{ijt}) left but also looking each period at the number of shortages incurred. For this model the cost of a shortage is immediately taken into account in the period that it has occurred since it is then directly back-ordered.

Contrastingly, the lost sales variant makes use of the variable s_{ij}, signifying the base-stock level or ordered quantity at the onset of each replenishment cycle, i.e., at period i. For this model the cost for holding quantities is considered each period (H_{ijt}), like in the back-ordering variant. However, the costs for the sales lost due to shortages are only considered and summed up at the end of a replenishment cycle so only H_{ijj-1} is important for finding the lost sale costs. Taken these differences into account the following MIP has been developed:

$$\text{minimize} \quad \sum_{i=1}^{N} \sum_{j=i+1}^{N+1} \left(K x_{ij} + \sum_{t=i}^{j-1} (h(s_{ij} - \mathbb{E}D_{it} x_{ij} + H_{ijt}) + \beta p H_{ijt}) + (1-\beta) v H_{ijj-1} \right)$$

subject to

$$\sum_{i=1}^{t-1} x_{it} = \sum_{j=t+1}^{N+1} x_{tj}, \qquad \forall t \in [2, N] \tag{13}$$

$$\sum_{j=2}^{N+1} x_{1j} = 1, \tag{14}$$

$$\sum_{i=1}^{N} x_{i,N+1} = 1, \tag{15}$$

$$s_{ij} = q_{ij} - \mathbb{E}D_{1i-1}, \qquad \forall i \in [1, N], j \in [i+1, N+1] \tag{16}$$

$$s_{ij} \leq M x_{ij}, \qquad \forall i \in [1, N], j \in [i+1, N+1] \tag{17}$$

$$q_{ij} \leq M x_{ij}, \qquad \forall i \in [1, N], j \in [i+1, N+1] \tag{18}$$

$$\sum_{i=1}^{t-1} (s_{it} - \mathbb{E}D_{it-1} x_{it} + H_{itt-1}) \leq \sum_{j=t+1}^{N+1} s_{tj}, \qquad \forall t \in [2, N] \tag{19}$$

$$H_{ijt} \geq a x_{ij} + b s_{ij}, \quad \forall i \in [1, N], j \in [i+1, N+1], (a, b) \in W_{it} \tag{20}$$

The hybrid model's objective function is primarily derived from the lost sales model's objective function. The first term, consistent in both models, computes the fixed cost. The second term aggregates holding costs over all periods and calculates the cost for backordering a fraction of β of the shortages for that period. The final term computes the cost associated with a fraction of $(1-\beta)$ of the shortages being treated as lost sales. To justify the assumption that β is known to the decision maker, we can refer to the work of Castellano (2015) as a basis. In their research, it is reasonably assumed that the value of β is known, and they utilize a specific value of 0.54 for this parameter. Similarly, in this study, we adopt this known value of 0.54 for β. By assuming a known value for β, we simplify the decision-making process and enhance the practicality of the model.

The first three constraints are identical to all other models and serve to ensure the proper functioning of the process. Constraint (16) establishes a connection between the decision variables of the two separate models. The base-stock level s_{ij} can be defined as $q_{ij} - \mathbb{E}[D_{1i-1}]$, this is also mentioned by Tunc et al. (2018). Constraints (17) and (18) ensure that the base-stock level (s_{ij}) and the expected cumulative order quantity (q_{ij}) can only take positive values if the replenishment cycle $[i, j)$ has been established. To maintain consistency, constraint (19) guarantees that the expected stock level at the end of a replenishment cycle does not exceed the base-stock level at the beginning of the next cycle. Due to this constraint and the connection made by constraint (16), original constraint (5) from the back-ordering model which ensured that the cumulative order quantities remain non-decreasing from one replenishment cycle to the next, is no longer necessary. Lastly, constraint (20) guarantees that the loss incurred at any period t is greater than or equal to the loss function evaluated at the base-stock level.

4 Methodology

Tunc et al. (2018) utilized Gurobi v6.5 as the MIP solver in their study. However, for the replication and extension of their work presented in this paper, CPLEX in Java will be employed as the solver. The experiments will be conducted on a MacBook equipped with an M1 processor and 8 GB of RAM. So it is expected that there might be some difference in the result due to the different hardware used, but it should not have a large disruptive impact. For the MIP models described earlier, we will employ the 11-piece lower bound approximation of the normal distribution loss function proposed by Rossi et al. (2014). The details of this approximation will be elaborated upon in the following section. It is important to note that all the data generated and used by Tunc et al. (2018) is readily available and will be used in our study. As a result, we anticipate obtaining almost identical results to those reported in their original research.

4.1 Linear approximation of the loss function

An important aspect of the model, of which the method is not extensively covered in the original paper, is the computation of the linear approximation of the loss function. It is briefly mentioned in the paper that 'for BM and PM, we use Rossi et al. (2014) 11-piece lower bound approximation of the normal distribution loss function' (Tunc et al., 2018) but further elaboration is not given. For comprehension and clarity it will be illustrated in this thesis as it took up a large part of the methodology of replication.

The optimization problem and MIP model at hand aims to minimize the cost or losses made. It is assumed that the demand has a known cumulative distribution function and first-order loss function. However, the loss function for the normal distribution (which is the one that is used) is rather complicated and does not have a closed-form. Hence a piecewise linear approximation will be used instead. We aim to have the set of pairs $W = \{(a_1, b_1), (a_2, b_2), ...\}$ where a_i represents the intercept and b_i the slope of a finite number of linear functions.

Rossi et al. (2014) have come up with a method to create successful piecewise linear upper and lowers bounds for normally distributed random variables that can be used for various applications. The linear approximation that they have created relies on a couple of constant parameters that stand independently from the mean and standard deviation of the normally distributed variables under consideration.

To develop their approximation, they used a 'minimax strategy'. Essentially, the idea is to divide the range over which the function is defined into several smaller partitions. Within each partition, the function is approximated by a straight line. The objective is to choose these straight lines in such a way that the maximum error between the lines and the actual function is minimized. For each partion they computed two quantities: p_i, which is the probability that a random variable of interest, ω, lies within that interval, and $E[\omega|\Omega_i]$, which is the expected value of random variable ω conditional on it being within that interval (Ω_i). These two parameters essentially determine the position and slope of the straight line that will approximate the curve in that sub-range. In their paper, Lemma 11 shows that each linear segment of the lower bound function is tangent to the actual complementary first-order loss function at a certain point. They then used Lemma 12 to demonstrate that the maximum approximation error between the actual complementary first-order loss function and the piecewise linear lower bound, occurs at a breakpoint. The authors used a system of non-linear equations to determine these breakpoints and optimized this system using the Gauss-Newton method. The result of this are the values that can be found in table 1 of their paper and which are used to calculate the piecewise linear approximation (Rossi et al., 2014).

A couple more steps are needed to get the desired set W. The paper writes all outcomes using the complementary loss function but the loss function is needed for our method. Lemma 3 is used to convert from the complementary first-order loss function to the normal first-order loss function. Then Lemma 10 is used, which shows how the lower bound presented in the paper is a piecewise linear function. This is used to create the set of intercepts and slopes.

Since the demand over multiple periods is needed, that is, from period i to t, we are dealing with the sum of independent normally distributed variables. Hence μ will be given as the sum of the expected demands from periods i to t and σ will be the sum of the variations. To get the correct σ, first the variance for each period is independently calculated. So the demand is multiplied with the variation coefficient and then squared and finally these are all added together to get the total variance. Then the intercept and slopes are computed with the recursive equations as shown below. Where the parameters $\mathbb{E}[\omega|\Omega_i]$ and p_i are as discussed above and are given in table 1 of Rossi et al. (2014), and can be found in appendix A.

$$
\begin{array}{ll}
\textit{Intercepts } (a_i) & \textit{Slopes } (b_i) \\
a_0 = \mu & b_0 = -1 \\
a_1 = a_0 + b_0(\sigma(\mathbb{E}[\omega|\Omega_1] + \mu) & b_1 = p_1 + b_0 \\
a_2 = a_1 + b_1(\sigma(\mathbb{E}[\omega|\Omega_2] - \mathbb{E}[\omega|\Omega_1]) & b_2 = p_2 + b_1 \\
\vdots & \vdots \\
a_i = a_{i-1} + b_{i-1}(\sigma(\mathbb{E}[\omega|\Omega_i] - \mathbb{E}[\omega|\Omega_{i-1}]) & b_i = p_i + b_{i-1} \\
\vdots & \vdots \\
a_{10} = a_9 + b_9(\sigma(\mathbb{E}[\omega|\Omega_{10}] - \mathbb{E}[\omega|\Omega_9]) & b_{10} = p_{10} + b_9
\end{array}
$$

4.2 Data generation

For their computational study, Tunc et al. (2018) have created two set instances, Set-A and Set-B, to test and evaluate their formulations under different conditions. Set-A is designed to assess how well the formulations perform under a variety of parameter settings. They consider various factors that could impact the effectiveness of the formulations:

- Holding cost (h) is constant at 1 for all instances.
- Three setup costs (K) are considered: 225, 900, 2,500.
- For the back-ordering model, service level models, and the lost sales models, they use the following respective parameters $p \in \{2, 5, 10\}, \alpha \in \{0.90, 0.95, 0.99\}, \beta \in \{0.80, 0.90, 0.95\}, \beta_c \in \{0.80, 0.90, 0.95\}$, and $v \in \{10, 20, 40\}$. Hence for our computational study only the values of p and v are relevant.
- Three planning horizon lengths are considered (N): 20, 30, 40.
- It is assumed that demands over the planning horizon are normally distributed, with three different coefficient of variation values $\rho \in \{0.1, 0.2, 0.3\}$.
- Mean demand values are randomly generated using two demand patterns: Erratic and Lumpy.
    - For Erratic, mean demands are drawn from a uniform distribution between 0 and 100. Erratic demand is marked by frequent and unpredictable fluctuations in demand

levels, with no discernible pattern or trend. This pattern is usually associated with rapidly changing customer preferences, for example in the tech industry.

  - For Lumpy, mean demands are drawn from a distribution that is 0 to 420 with a 20% probability and 0 to 20 with an 80% probability. Lumpy demand is characterized by infrequent and irregular spikes in demand interspersed with periods of little to no demand. This pattern is common in products with seasonal demand.

For each combination of input variable, 10 random problem instances are created, leading to a test set of 8100 test instances.

Set-B, in contrast, is designed to assess how well the formulations can scale. This means testing the formulas with larger problem instances:

- The problem will be considered with instances with longer planning horizon lengths (N): 50, 60, 70, 80, 90, 100.
- An erratic demand pattern is assumed for all instances with a coefficient of variation of 0.3.
- For this 10 random instances are generated for each planning horizon length, leading to a total of 300 problem instances.

It is important to note that the purpose of Tunc et al. (2018)'s computational study was to compare and demonstrate the efficiency of the extended formulation, both with and without the dynamic cut generation approach. However, this thesis focuses on replicating the extended formulation without employing the dynamic cut generation approach and then utilizes these results to develop an extended hybrid version of the model. Consequently, the results are presented in a manner that slightly deviates from how they are shown in Tunc et al. (2018).

The various instances discussed above have been employed and subsequently compared across all three models. Specifically, for the back-order and lost sales models, the results are benchmarked against the computational findings of Tunc et al. (2018). Moreover, these instances are also utilized for the evaluation of the newly devised hybrid extension. For this variant, a β value of 0.54 is adopted, as previously mentioned. Since this hybrid approach was not part of the original paper, a direct comparison is not feasible. However, it is anticipated that the performance of the hybrid model falls within the range defined by the results of the original two models.

4.3 Simulation

To validate the findings obtained from three distinct MIPs corresponding to three different variants, a set of three simulation models is developed. The results derived from the MIPs are used to check if the simulation and MIP objective values match. More specifically, the replenishment cycles and the associated base-stock levels or cumulative order-up-to levels, are used and serve as the input parameters for simulating the inventory process.

In every iteration, the simulation mimics the real-life inventory process by calculating the expected demand based on the normal distribution, using the provided mean demand and variation coefficients. The simulation incorporates the process of the replenishment cycles and keeps track of the levels of losses, holdings and replenishment. Ultimately, the simulation calculates the average total costs for each scenario. The primary motive behind this extra validation step is to ensure the feasibility of the results given by the newly developed hybrid model, as the two other models can be compared to the results from the original paper. The simulations will be programmed in Java. For the simulation 100,000 iterations will be run. Since the simulation has a very low computational effort and takes a short time it can be run many times to get a more accurate results. This is useful especially as, with the given parameters, the variation of the results can become quite large.

First the two original models will be simulated, the back-order and lost sales models. Subsequently, after it has been checked that these work properly, they will be combined to create a simulation for the hybrid variant.

5 Results

5.1 Results Replication MIP - Lost Sales & Back-order

Below are the results obtained for a selection of instances that were replicated for the back-order and lost sales models. The tables present a variety of results for both set A and set B, showcasing different parameter values (K, p, ρ), as well as demand variations. While not all results are included to avoid redundancy, a representative group is provided, highlighting the variations between parameters and demands.

For clarity, consistency, and the ability to compare results across tables, the expected demand patterns used when referring to set A with the lumpy demand pattern will be explicitly mentioned. Other corresponding values will be displayed in the tables. Please note that the tables include the objective value found during replication (Obj) and the objective value reported in the original paper (Obj (P)). The same goes for the running time, except with Rel. Diff. Obj indicating the relative difference between the replicated outcome and the value reported in the original paper, instead of the absolute difference.

The following expected demand patterns are used in the paper whenever results from set A with the lumpy demand pattern are referenced. Any other relevant values will be displayed alongside the tables:

$$D_1 = \{7, 17, 16, 5, 2, 297, 8, 19, 4, 5, 5, 372, 19, 19, 88, 10, 7, 13, 156, 12\}$$
$$D_2 = \{10, 1, 11, 10, 331, 17, 14, 3, 11, 16, 15, 7, 20, 12, 4, 19, 20, 3, 13, 11\}$$
$$D_3 = \{20, 18, 13, 14, 14, 18, 20, 2, 17, 67, 13, 8, 7, 16, 20, 13, 198, 17, 6, 11\}$$

5.1.1 Replication Results Set A: Lumpy Pattern

Table 1 and table 2 show the output given by the MIP when using the demand pattern and parameters from Set A (and the expected demands D_1, D_2 and D_3) as mentioned before, for the back-ordering and lost variant respectively. For this section the generating pattern that is focused on is lumpy. Though set A also contains demands from the erratic pattern, since these are also applied in set B they will be focused on in that section.

Table 1: Back-order variant, Lumpy, Set A

K	p	ρ	D	Obj	Run Time	Obj (P)	Run Time (P)	Diff. Obj	Rel. Diff. Run Time
225	2	0.1	D_1	1643.1780	1.5024	1643.1785	0.3438	-0.0006	3.3700
900	2	0.1	D_1	4213.4502	0.3456	4213.4507	0.3125	-0.0006	0.1060
2500	2	0.1	D_1	8131.8740	0.5466	8131.8744	0.2969	-0.0004	0.8425
225	2	0.1	D_2	1344.4926	0.4627	1344.4930	0.3907	-0.0004	0.1851
225	2	0.2	D_2	1474.8223	0.5873	1474.8224	0.4063	-0.0001	0.4437
225	2	0.3	D_2	1527.8183	0.4008	1527.8185	0.4532	-0.0002	-0.1124
225	2	0.1	D_3	1397.7894	0.6809	1397.7896	0.3486	-0.0001	0.9555
225	5	0.1	D_3	1560.0564	0.4049	1560.0568	0.2705	-0.0004	0.4982
225	10	0.1	D_3	1634.1281	0.4753	1634.1287	0.2188	-0.0006	1.1721

Table 2: Lost sales variant, Lumpy, Set A

K	v	ρ	D	Obj	Run time	Obj (P)	Run time (P)	Diff. Obj	Rel. Diff. Run Time
225	10	0.1	D_1	1816.0539	0.6906	1816.0546	0.2031	-0.0007	2.4003
900	10	0.1	D_1	4656.1838	0.6079	4656.1845	0.2344	-0.0006	1.5959
2500	10	0.1	D_1	8789.5575	0.3973	8789.5577	0.2656	-0.0002	0.4963
225	10	0.1	D_2	1511.0675	0.4739	1511.0678	0.3016	-0.0003	0.5717
225	10	0.2	D_2	1707.8628	0.4804	1707.8698	0.3125	-0.0070	0.5384
225	10	0.3	D_2	1921.3347	1.7728	1921.3354	0.3125	-0.0007	4.6681
225	10	0.1	D_3	1614.9225	0.3798	1614.9227	0.2738	-0.0002	0.3860
225	20	0.1	D_3	1680.6914	0.7880	1680.6918	0.2705	-0.0005	1.9130
225	40	0.1	D_3	1735.3044	0.5637	1735.3055	0.2656	-0.0011	1.1235

From both tables 1 and 2 it is quite clear that regardless of the demand pattern used or parameters used, the objective values obtained during replication are very close to the ones reported in the original paper (Obj (P)). The relative differences in objective values (Diff. Obj) are small, mostly within the magnitude of 10^{-4}, indicating that the replication process was able to closely follow the results of the original paper.

The running time, however, varies more considerably. In the first row, for example, the replicated running time is approximately 3.37 times the running time reported in the original paper. This discrepancy in running time might be due to different hardware or software environments used for the replication compared to the original paper. The relative difference in running time seems to be more varied in the lost sales variant compared to the back-order variant. This might suggest that the computational complexity or the sensitivity of the lost sales model to specific parameters and demand patterns is different from the back-order model.

5.1.2 Replication Results Set B: Erratic Pattern

Table 3 and table 4 show the output given by the MIP when using the demand pattern and parameters from Set B. This set has larger problem instances with longer planning horizon lengths (N). Unlike Set A, Set B features a larger instance structure, hence with a larger range of planning horizon lengths ranging from periods of 50 up to 100. The expected demands corresponding to Set B are derived from an erratic pattern, and all first instances are used to achieved the results shown below. A detailed list of the expected demands used for these tables can be found in Appendix **??**. It is important to note that the demand sets themselves are not displayed within the paper due to the extensive nature of the planning horizons in Set B. Instead, the results from the first instance of each planning horizon length are included to provide an indicative overview of the outcomes across different lengths of planning horizons.

Table 3: Back-order variant, Erratic, Set B

K	p	ρ	N	Obj	Run Time	Obj (P)	Run Time (P)	Diff. Obj	Rel. Diff. Run Time
225	10	0.3	50	10895.1450	13.5690	10895.1576	11.9218	-0.0126	0.1380
225	10	0.3	60	11986.6975	31.6455	11986.7093	20.8124	-0.0117	0.5226
225	10	0.3	70	15088.1200	29.1807	15088.1384	51.3592	-0.0185	-0.4300
225	10	0.3	80	16890.7069	41.7765	16890.7267	72.4060	-0.0198	-0.4243
225	10	0.3	90	19190.0310	85.8830	19190.0509	272.3272	-0.0199	-0.6856
225	10	0.3	100	20947.3730	277.4619	20947.3932	335.2020	-0.0202	-0.1721

Table 4: Lost sales variant, Erratic, Set B

K	v	ρ	N	Obj	Run time	Obj (P)	Run time (P)	Diff. Obj	Rel. Diff. Run Time
225	40	0.3	50	12164.9438	33.6251	12164.9795	7.2500	-0.0357	3.6294
225	40	0.3	60	13245.5262	25.3702	13245.7862	20.1093	-0.2600	0.2620
225	40	0.3	70	16872.0795	47.9613	16874.0382	37.2811	-1.9587	0.2861
225	40	0.3	80	18774.5421	58.6102	18774.5964	55.5623	-0.0543	0.0548
225	40	0.3	90	21388.2425	89.1557	21388.3053	86.1716	-0.0628	0.0346
225	40	0.3	100	23414.2975	143.8755	23414.7472	113.1715	-0.4497	0.2703

Similar to what was mentioned in Section 5.1.1, the results obtained using the expected erratic demand pattern and parameters from Set B are closely aligned with those presented in the original paper. However, there is an apparent outlier, when $N = 70$, where the difference between the results is noticeably larger than in other cases. This might be because of a random calculation error. It is also noteworthy that the discrepancies in results for the erratic demand pattern are slightly larger compared to those for the lumpy pattern. This could be blamed on the fact that the erratic pattern involves a larger set, where even minor deviations can be amplified and show up as large errors. The important take away from this replication, as mentioned by Tunc et al. (2018) is that the extended model can solve realistic-sized instances in reasonable computational times, even without the dynamic cut generation approach. Overall, both models have been successfully replicated.

5.2 Results for the Hybrid MIP Model

Below in table 5 the results for the hybrid MIP model can be found. In this table again the expected demand patterns D_1, D_2 and D_3 are used. The objective value that was found by the hybrid model can be seen in the sixth column, and the two columns next to it display the objective values of the back-order and lost sales model when using the same parameters (p as the penalty parameter for back-ordering and v as the penalty parameter for lost sales).

From Table 5, we can observe that for a given set of parameters, the hybrid model generally yields objective values that are inbetween the values given by the back-order and lost sales models. This was expected and shows the hybrid model's ability to balance the trade-offs between back-ordering and lost sales by incorporating features from both models. It seems that often the Hybrid model yields an objective value that is closer to the of the back-ordering model. This is also expected as the β value is taken to be 0.54, so a slightly higher fraction of the unsatisfied demand is back-ordered in comparison to the fraction that is lost.

It is also interesting to see that when the penalty costs are the same for the models (i.e. $p = 10$ and $v = 10$) then there is still a difference, though small (about 20), between the objective values. This can be attributed to the fact that for the back-ordering model, costs can be incurred during the periods and not only as a sum afterwards like in the lost sales. Hence the back-ordering model might be more focused on the losses of each period individually. This instance, where $p = v = 10$ supports and shows that the models have different work methods, and that difference are not only caused by differences in penalty costs. The hybrid model finds, also for this case, a solution situated in between the back-order and lost sales solutions.

Table 5: Hybrid variant, Lumpy, Set A

k	p	v	ρ	D	obj. Hybrid (MIP)	obj. Back-order (MIP)	obj. Lost Sales (MIP)
225	2	10	0.1	D_1	1751.091231	1643.17796	1816.054601
900	2	10	0.1	D_1	4579.423181	4213.450157	4656.184475
2500	2	10	0.1	D_1	8581.459793	8131.874005	8789.557681
225	2	10	0.1	D_2	1468.361752	1344.492559	1511.067834
225	2	10	0.2	D_2	1632.601613	1474.822295	1707.869783
225	2	10	0.3	D_2	1707.564648	1527.818294	1921.33544
225	2	10	0.1	D_3	1559.565785	1397.789441	1614.922703
225	5	40	0.1	D_3	1692.396433	1560.056816	1735.305491
225	5	20	0.1	D_3	1642.963645	1560.056442	1680.691842
225	10	10	0.1	D_3	1626.03342	1614.922703	1634.128699
225	10	20	0.1	D_3	1661.22106	1634.128699	1680.691842
225	10	40	0.1	D_3	1704.052692	1634.128074	1735.305491

In table 6, we present results for the hybrid model under erratic demand patterns (Set B). Analyzing the second table, we can observe an increasing trend in the objective values of the hybrid model and run time as the number of periods (N) increases. This suggests, as expected that as the complexity of the problem increases (with more periods to consider), the hybrid model takes longer to find an optimal solution and the objective value becomes higher. Also, as expected, the hybrid model's objective values continue to fall between those of the back-order and lost sales models. The run time indicates the computational efficiency of the hybrid model, which appears to increase non-linearly with the number of periods. Which again is expected as the number of feasible replenishment cycles increases non-linearly with the number of periods.

Table 6: Hybrid Variant, Erratic, Set B

K	p	v	ρ	N	Obj. Hybrid	Obj. Backlog	Obj. Lost Sales	Run Time
225	10	40	0.3	50	11761.4	10895.145	12164.94379	11.02353
225	10	40	0.3	60	12843.43878	11986.69754	13245.52618	24.222049
225	10	40	0.3	70	16261.77496	15088.11995	16872.07953	48.17682099
225	10	40	0.3	80	18151.55168	16890.70687	18774.54205	66.37958598
225	10	40	0.3	90	20678.60146	19190.031	21388.2425	120.441967
225	10	40	0.3	100	22703.00633	20947.37298	23414.29749	198.537636

5.3 Results for the simulation

Similarly as is done in the paper by Tunc et al. (2018), to check whether the outputs of the MIP models make sense and are feasible, the developed inventory policy is simulated to see what objective value, or total cost the simulation gives. The comparison of the MIP output and the result of the simulation is useful for validating the robustness and accuracy of the MIP model, as simulation is generally considered a reliable method for mimicking real-world scenarios. To do so the replenishment cycles and their corresponding order quantities are inputted into the simulation which then mimics the status of the inventory over the time planning horizon. Firstly, as can be found below in table 7, the back-order and lost sales models are simulated to check whether the simulation and MIP results match.

Table 7 shows that the results of the simulation and the MIP are quite closely algined. However, it is essential to note that there are some differences between the two sets of results, some larger than other. These differences are mostly like due to the inherent variability and randomness of the simulation. The results are close enough to confidently say that the simulation correctly mimics the process that the MIP is trying to optimise.

Table 7: Comparison Simulation and MIP Back-order and Lost sales variants

k	p	v	ρ	D	obj. Back-order (Simulation)	obj. Back-order (MIP)	obj. Lost Sales (Simulation)	obj. Lost Sales (MIP)
225	2	10	0.1	D_1	1642.476	1643.178	1816.130	1816.055
900	2	10	0.1	D_1	4218.436	4213.450	4661.668	4656.184
2500	2	10	0.1	D_1	8143.599	8131.874	8794.807	8789.558
225	2	10	0.1	D_2	1324.40	1344.493	1515.118	1511.068
225	2	10	0.2	D_2	1489.349	1474.822	1709.112	1707.870
225	2	10	0.3	D_2	1538.238	1527.818	1939.007	1921.336
225	2	10	0.1	D_3	1402.983	1397.789	1615.220	1614.923
225	5	20	0.1	D_3	1574.345	1560.056	1684.184	1680.692
225	10	40	0.1	D_3	1630.250	1634.128	1740.362	1735.306

The simulation processes for both the back-ordering and lost sales variants have been integrated to create a comprehensive simulation that mimcs the behavior of the process when applying the hybrid model. This integration is a crucial step as it serves as the most effective means to evaluate whether the developed hybrid variant is both feasible and applicable in real-world scenarios. Table 8 presents a comparison between the objective values obtained through the MIP model and the total costs produced by the simulation. This comparison validates the practicality and reliability of the hybrid inventory model.

Table 8: Comparison Simulation Results Hybrid variant

k	p	v	ρ	D	obj. Hybrid (Simulation)	obj. Hybrid (MIP)
225	2	10	0.1	D_1	1751.300	1751.091
900	2	10	0.1	D_1	4605.137	4579.423
2500	2	10	0.1	D_1	8535.787	8581.460
225	2	10	0.1	D_2	1480.802	1468.362
225	2	10	0.2	D_2	1615.559	1632.602
225	2	10	0.3	D_2	1698.592	1707.565
225	2	10	0.1	D_3	1543.267	1559.566
225	5	20	0.1	D_3	1635.466	1642.964
225	10	40	0.1	D_3	1698.230	1704.053

At a first glance it is clear that there exists a close alignment between the values generated by the simulation and those produced by the MIP model. This close alignment is an indication that the hybrid inventory model is behaving consistently, and suggests that the results are reliable. However, it is worth noting that there are minor deviations between the values from the simulation and those from the MIP model. For example, in the second row for D_1, the objective values are 4605,13 and 4579,42 for the simulation and MIP respectively, which means there is a difference of 25 which is rather large. These slight differences can be attributed to factors such as approximation errors or intrinsic differences in the computational methods between the simulation and the MIP model. Another cause might be that there is a small calculation error somewhere in the methodology which is highlighted only when certain parameters take on certain values.

Additionally, the MIP and simulation results show consistent trends in response to the variation of parameters. A notable example is the upward trend in values for both simulation and MIP as ρ increases from 0.1 to 0.3 in the data corresponding to D_2. This coherence demonstrates both the accuracy of the simulation and the MIP in modeling the correct behaviour.

Taking the close alignment, minor deviations, and consistent trends into consideration, Table 8 substantiates the credibility and applicability of the hybrid inventory model in practical scenarios. Decision-makers can utilize this model with confidence for making inventory decisions, as it has demonstrated that its results are not only theoretically sound, as shown by the MIP model, but also practically viable, as validated by the simulation.

6 Conclusion

In this thesis we have addressed and aimed to replicate a solution to the stochastic lot-sizing problem, namely through using the static dynamic uncertainty strategy and the extended mixed-integer programming formulation that was suggested by Tunc et al. (2018). The extended model for the static dynamic uncertainty strategy turns out to be both computationally efficient and flexible. This formulation can handle variants of the problem characterized by various parameters such as penalty costs, service level constraints, backorders, and lost sales. Addressing the stochastic lot-sizing problem is crucial in modern business operations. It enhances decision-making by integrating uncertainty and unpredictability, leading to more effective choices regarding demand and inventory levels. The main contribution of our work was to replicate and build on the suggested formulation. We have proposed a simple hybrid model that allows the process to be optimized while incorporating both back-orders and lost sales. This hybrid model aims to better reflect real-world scenarios where customers may choose to wait for back-ordered items or seek alternative options.

Firstly, the paper successfully replicated the findings of Tunc et al. (2018). Through comparison with the original paper and additional comparison with the help of a simulation it was clear that the models, both the back-order and lost sales model, gave the expected output and affirmed that the replication process had been reliable and accurate. Once these were replicated properly, we combined them to create the hybrid model. This was again compared with both existing models and with a simulation. The solutions from the hybrid MIP and that of the simulation aligned closely. There were some minor differences, but these most likely can be attributed to the inherent randomness of the simulation process. Upon reflection, the use of both the mathematical programs and the simulation allowed for a robust research process. Due to this cross-validation there is a higher degree of confidence in the reliability of the results. Though our results proved to properly replicate those of Tunc et al. (2018) it is important to mention the potential errors in their paper. Despite its computational efficiency, the extended formulation as developed in their paper, is not exact due to persistent systematic errors, particularly under specific parameter settings.

Regarding future work on the thesis topic, there are several recommendations to consider. Firstly, as Tunc et al. (2018) mention, future work should focus on developing exact methods for the static-dynamic uncertainty strategy, and on analysing the problem with limited distributional information on random demands, as this assumption becomes more and more restrictive over longer planning horizon lengths. Furthermore, regarding the content of this thesis, it would be beneficial to explore further extensions and refinements of the proposed hybrid model to enhance the model's performance and applicability in various real-world scenarios. For instance, instead of assuming a fixed ratio of back-orders and lost sales, as we did based on the findings of Castellano (2015), it would be more realistic to treat this ratio as a random variable. By incorporating this variability, the solution becomes more complex but also more representative of real-world situations. Additionally, it might be worth considering the development of heuristics as an alternative to solving for optimality. Exploring this approach could provide valuable insights and potentially offer practical instead of fully optimal solutions.

The extended formulation and hybrid model's applicability to other inventory management problems and various industries may potentially lead to more contributions to the field. Utilizing these models in practical settings could unveil additional factors, constraints, lead times, or pricing considerations that need to be addressed. Further developing on this idea and taking the hybrid model into consideration, in real-world scenarios might lead to even more extensions. Adding in service level constraints has not been considered in this thesis but is relevant to incorporate when expanding the scope and complexity of the models.

In summary, this thesis has successfully replicated and extended the formulation proposed by Tunc et al. (2018). In terms of contributions to the field, we have introduced the combination of the static-dynamic uncertainty strategy and a hybrid model taking both back-orders and lost sales into account. The replication and simulation results validate the effectiveness and accuracy of the models. Furthermore, the findings highlight the importance of considering uncertainty and integrating dynamic decision-making components in inventory policies, with many possibilities for future extensions.

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A Parameters used for the Piecewise Linear Approximation

Parameters taken from Rossi et al. (2014).

Table 9: Values for Parameters used for approximation

p_i	$E[\omega\|\Omega_i]$
0	-2.5
0.0420611	-2.13399
0.0836356	-1.39768
0.110743	-0.9182
0.127682	-0.526575
0.135878	-0.17199
0.135878	0.17199
0.127682	0.526575
0.110743	0.9182
0.0836356	1.39768
0.0420611	2.13399
0	2.9

B Mean Demands used for Set B Results Tables

$N = 50 : \{97, 134, 124, 140, 181, 151, 33, 45, 91, 79, 193, 146, 94, 71, 46, 121, 185, 34, 91, 69, 122, 51,$
14, 136, 62, 130, 182, 117, 183, 161, 13, 50, 133, 96, 1, 165, 130, 87, 105, 35, 111, 29,
133, 189, 44, 54, 116, 176, 34, 79}
$N = 60 : \{22, 53, 36, 135, 197, 49, 6, 150, 192, 103, 14, 176, 105, 34, 25, 18, 50, 105, 121, 67, 173, 159, 1,$
$129, 19, 81, 85, 66, 25, 87, 2, 23, 85, 88, 36, 48, 24, 103, 92, 122, 191, 167, 13, 11, 38, 3, 143, 91, 110, 84, 126,$
157, 77, 125, 99, 195, 134, 75, 103, 181}
$N = 70 : \{136, 178, 105, 119, 60, 182, 26, 105, 2, 57, 83, 60, 198, 135, 9, 122, 150, 144, 31, 144, 187,$
186, 163, 142, 12, 23, 28, 160, 123, 76, 33, 80, 26, 159, 5, 161, 97, 175, 69, 140, 86, 168,
14, 193, 106, 158, 78, 148, 163, 139, 146, 4, 42, 123, 48, 172, 143, 192, 13, 188, 144, 10,
8, 85, 33, 13, 141, 122, 199, 62}
$N = 80 : \{62, 25, 120, 82, 173, 44, 16, 168, 96, 108, 188, 122, 105, 141, 21, 53, 106, 53, 35, 103, 78, 19, 88,$
195, 87, 16, 154, 56, 76, 144, 5, 132, 29, 30, 100, 117, 173, 152, 179, 146, 50, 86, 81,
54, 75, 143, 160, 101, 63, 145, 66, 183, 56, 74, 26, 111, 130, 6, 153, 98, 178, 134, 110,
8, 163, 126, 71, 126, 7, 123, 84, 193, 33, 150, 6, 86, 76, 164, 151, 70}
$N = 90 : \{49, 120, 87, 171, 75, 115, 14, 4, 63, 184, 172, 76, 151, 105, 162, 61, 85, 63, 114, 173, 5, 113,$
$139, 50, 101, 135, 73, 28, 107, 159, 59, 87, 30, 46, 102, 11, 163, 187, 74, 27, 53, 159, 121, 103, 128, 64,$

146, 179, 91, 92, 73, 86, 170, 120, 61, 52, 94, 78, 112, 91, 186, 98, 191, 196, 56, 162, 169, 89, 91, 8, 186, 17, 134, 32, 64, 61, 133, 60, 93, 164, 139, 45, 108, 18, 23, 28, 21, 171, 180, 29}

$N = 100 : \{34, 102, 55, 161, 78, 177, 27, 20, 21, 182, 36, 63, 65, 111, 24, 58, 155, 84, 89, 114, 159, 131,$ 37, 72, 156, 178, 111, 23, 68, 3, 140, 48, 74, 173, 181, 43, 31, 122, 11, 36, 131, 175, 25, 83, 157, 135, 153, 62, 165, 68, 44, 55, 4, 22, 96, 102, 80, 188, 178, 130, 30, 157, 173, 49, 47, 170, 60, 90, 42, 77, 170, 152, 22, 110, 181, 68, 65, 136, 92, 51, 23, 107, 93, 162, 179, 181, 129, 66, 94, 59, 120, 116, 25, 38, 81, 122, 197, 144, 82, 48}